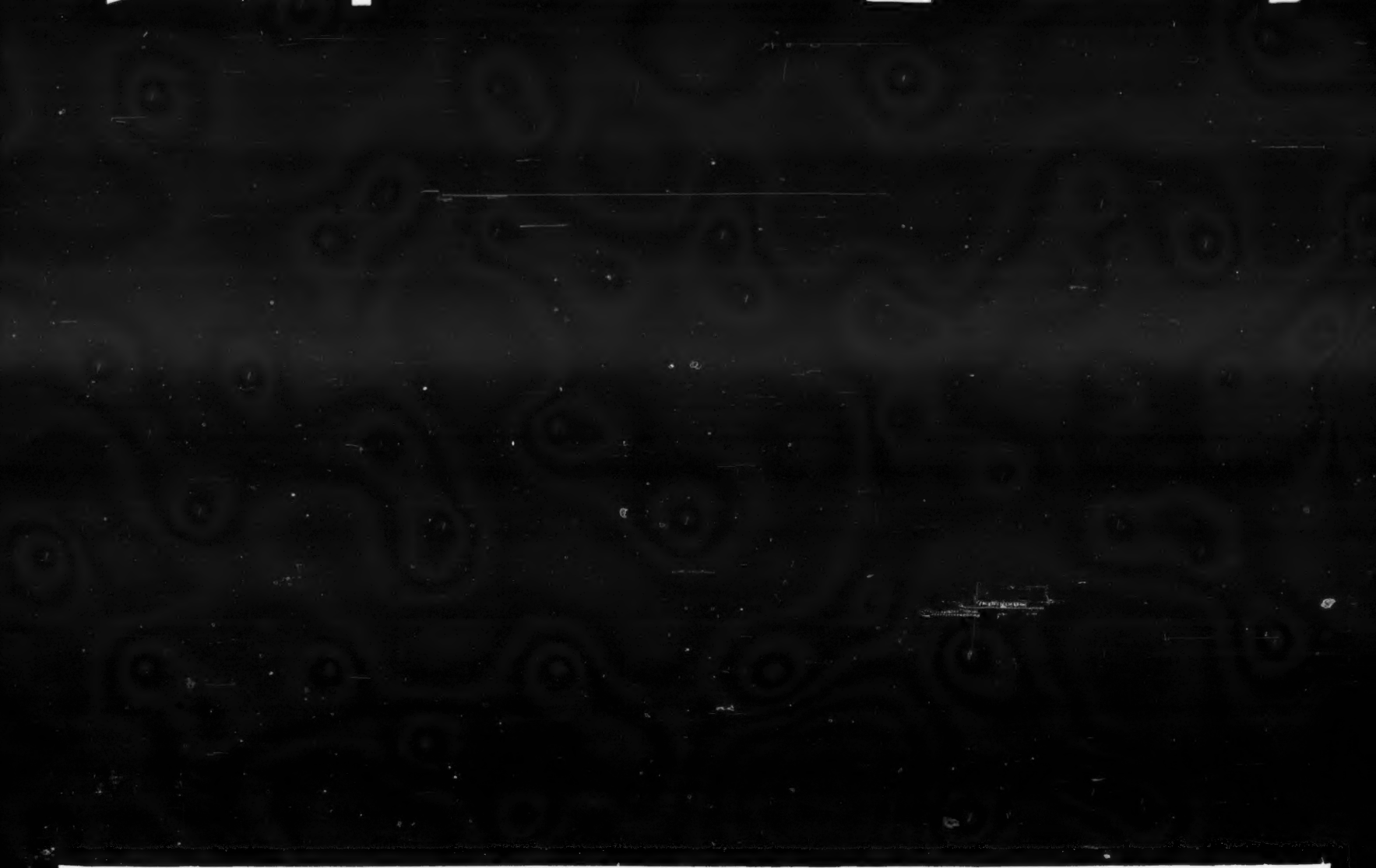

THE SCHOLAR IN A SCIENTIFIC WORLD*

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1. Introduction. Recorded history began with the scholar. Sometime in that great blank period from twenty to six thousand years ago, the art of writing was developed so that the scholar could record the legends and history of his nation. The scholar was then also the priest and the lawyer and the physician as he has been throughout most of recorded history.

Since earliest times scholars have lived in communities. These communities or monasteries, or musea, were the early colleges and universities. They have constituted the chief vehicle in the development of civilization. If these little bands of scholars had not existed, we should now be in the woods carrying stone axes.

A nation is just about as great as its universities. They are the ganglia in the central nervous system of the nation, whence come the nerves that stimulate its intellectual, industrial, and political life. Without the leadership of the universities, national life would probably continue for a while by inertia but would gradually slow down and succumb to the competition of rival nations.

The function of the university as a teaching institution is purely incidental. Its principal function is to keep alive the great wealth of knowledge and culture that past ages have collected, and to add to that wealth through the encouragement of scientific research and creative art. Knowledge and art can be kept alive only by the creation of scholars, and the scientific horizon can be extended only by the creation of scientists. To create scientists and scholars is the primary function of the university.

This statement of the function of the university is orthodox. It is as old as the universities themselves. It is the idea which induced Alexander to establish the Museum at Alexandria, the institution which shone out over the ancient world for the 600 years of its highest culture and where much of our mathematics was made. Alexandria was doubtless the greatest university of all time, and the burning of its library in 389 might well mark the beginning of the Dark Ages.

The Renaissance was marked by, and was in a very large measure due to, the establishment of the great universities in Padua, Paris, and Oxford, and soon in many other places. The study of the great legacy of Greece was brought into Europe by these universities, where it had been extinct for many years. Men had to learn again that it is not evil to think, and that we live in a reasonable world. The scientific developments of our generation could not have taken place if these medieval universities had not come into being.

2. Science in Modern America. American universities have progressed amazingly in a century, and so has the country. From the position of a backward nation we have come to the very front. The opportunity to establish undisputed

* Retiring address as President of the Mathematical Association of America, Athens, Georgia, January 1, 1948.

intellectual leadership is ours. Whether we can maintain this position when the influx of German scientists stops remains to be seen.

As time goes on, and more and more is known of science, archaeology, and history, it becomes increasingly difficult to keep some branches of our culture from becoming lost. The day has long since passed when one man could have a working knowledge of all branches of science. The goal of the philosopher to know all things and to correlate them into one harmonious whole now seems very difficult of fulfillment.

The last war has destroyed many of the cultural links between our age and the past, particularly by the wanton bombing of the libraries and museums by both sides. It has also reduced the number of scholars in Europe so that some links that we had with our past are probably now gone forever. We must be willing to support scholars in all fields without everlastingly measuring their output in terms of American dollars.

Everything that I have said so far is trite and generally admitted, but we in the United States do not make it part of our practical thinking. We do not believe that the production of scholars is in itself a worthy objective, nor that the scholar is worthy of his hire. Our colleges and universities are supported by state and private funds, but the stated objective is the instruction of vast numbers of young men and women in ways and means of earning a better living than their fellows. They are taught citizenship and the American way of life. I am not always clear just what this means. Occasionally it seems to be football and the cult of the gods and goddesses of Hollywood.

We are now living in the Age of Science. One might almost say that Science has become our national religion. It has revolutionized our daily lives, our knowledge of the past, and our hopes for the future. Its miracles are commonplace and we can perform them ourselves. The researchers in medicine are the angels of mercy, and in popular imagination the Atomic Scientists are the devils.

This elevation of Science to a god-like eminence creates many difficult problems for the scientist, who is not primarily interested in becoming a high priest. First, everyone wants to be a scientist. Astrologers, numerologists, even religious groups, conjure with the name "Science." On a little higher plane, we have Political Science and the Science of Economics, when everyone knows that these subjects are not yet truly scientific. Even Psychology has renounced its parent, Philosophy, and wishes to be known as a science.

Everyone is pathetically eager to understand science, and of course most persons are unable or unwilling to pay the price in long years of study. Our book stores are flooded with books on "Relativity in Five Easy Lessons," illustrated with doodles, and "How to Make an Atomic Bomb." Even the elementary schools have courses in General Science, and eight-year-olds lisp that Einstein discovered the relativity of time and space. It is questionable whether sophistry at this level does more harm than good, but probably nothing can be done about it. It is part of the ritual in the worship of Science. A classic example is the quotation from the school boy's paper: "Gravitation was discovered by Sir Isaac

Walton while he was digging for hook-worms under an apple tree. It is more noticeable in the fall than in the spring."

If it were necessary or desirable to draw a fine distinction between the scholar on the one hand, and the artist or scientist on the other, it would have to be that the scholar is primarily interested in perpetuating the known, the scientist in new creation. But they are not independent. New discoveries are now rarely made by attic inventors. The creator is so steeped in the knowledge of his predecessors that he can begin where they left off. Otherwise he merely rediscovers.

There are fashions and fads in intellectual pursuits as well as in politics or clothes. At the time of Socrates in Greece, speculative philosophy was the fashion, and so was sculpture in stone. In the Middle Ages about the only outlet was theology. There were later periods in which great music was produced; others were notable for great paintings. In sixteenth century Italy there was a remarkable flowering of painting and sculpture. It must have been true that the people of this period were enthusiastic about art, and that the great painters were honored and important people. The result was that the best minds of the period were attracted into this field, and that they were stimulated by knowing that their work was appreciated.

In this period lived Leonardo da Vinci, one of the greatest potential scientists of all time. It has been said that if he had published his scientific discoveries, science would have been revolutionized. But he could not have published, for he had no sympathetic audience. In his own period he was known and honored as an artist, and it is only now in the Age of Science that his scientific thoughts are appreciated. If he had lived today, it is probable that he would have become a great scientist and that his potentialities as an artist would have remained undeveloped.

Deans of graduate schools and those who have served on university committees of award for scholarships will testify that, by and large, the undergraduate records of those who apply for fellowships in mathematics and the natural sciences excel all others. It is all very well to argue that we need the best minds in Sociology and Politics. The fact remains that the country is worshipping at the shrine of the Natural Sciences.

3. Science in the curriculum. There is now a widespread movement in the colleges and universities to incorporate some science into the liberal arts program. It seems as if all the weight of logic is in favor of this. We study and admire the gems of various older cultures, their stories, poems, songs, pictures, statues, and philosophical concepts. Included in the education of the well-rounded scholar should be the gems of thought of modern man. These gems are largely the processes of thought of the modern scientist.

But it is not going to be easy to design a curriculum which includes little dabs of the sciences without creating a nation of sophomores. This is now being done to some extent by the courses in General Science in the grade schools where students learn that Newton discovered gravitation by being bonked on

the head by an apple. But this does no lasting harm because the better students unlearn the fallacies and the poorer ones forget them. But to create a nation of college graduates who believe they know all about science from a Basic Course in the Physical and Biological Sciences would be a pity.

There is of course the classic instance of the sweet girl undergraduate who sat next to a famous astronomer at a Boston dinner party. She asked him what his business was. "I study astronomy," was the reply. "Oh, you do!", the girl answered in surprise, "Why, I finished that last year!"

Let us grant that it is no longer possible to deny the natural sciences their rightful place in the education of the humanist. There remains the problem of what science shall be presented, how it shall be presented, and by whom.

Clearly the student of the arts is not going to get a very complete course in any one scientific subject under this plan. Perhaps he isn't going to get any laboratory work at all. He may get nothing but a huge mass of undigested and undigestible facts. If the instructor is an experimental scientist, and is intent upon making a chemist or biologist of the student in one semester, it is not difficult to imagine a course which would be a veritable monstrosity, capable of inducing a permanent allergy on the part of the student victim toward all science.

Equally unsatisfactory will be the other extreme where the instructor is a philosopher who knoweth all things, believeth all things, and seeth through a glass darkly. He can clothe the simplest concept in metaphysical terms, which mean nothing to anybody. The flights of fancy which such people have indulged in, using Relativity and the Heisenberg Principle as justification, are known to all of us.

If science is to be taught in the same spirit as the other humanities, it will be taught with the object of exhibiting the thought processes of the scientist. One currently studies Art Appreciation without attempting to model in clay, and the liberal arts student can gain an insight into modern science without acquiring sulfuric acid burns.

The study of the scientific method, then, is the proper objective for a liberal arts course in the sciences. But the scientific method is the mathematical method, for the discoveries in science are obtained either by a direct use of known mathematics, or by the same methods of careful logical thinking to which mathematics is the best introduction. Without the scientific method, science becomes mere classification and memory.

Physics is now riding a crest of popularity not unmixed with awe. It will have a prominent place in the program. The lecturer will have the choice of merely listing the triumphs of Physics, or pulling the curtain a little and showing the student how these results were arrived at. Pulling the curtain means using mathematics.

Astronomy is another science which deserves a prominent place in any cultural course. Unfortunately that seems to be its only place at present, for our industrialists have not yet found a means of developing the natural resources of

the stars.

Perhaps I have given an unnecessarily long preamble to a mathematical audience before stating my point, which is this: One of the brightest gems in the crown of achievement of civilized man is the Differential and Integral Calculus, and no man has the right to call himself a Humanist if he is completely ignorant of its meaning.

Many members of the Mathematical Association of America are members of the curriculum committees of their respective institutions. It is to be hoped that they will have enough appreciation of their own subject, and enough knowledge of the fundamental role which it plays in modern Science, to demand that an adequate exposition of the mathematical method be incorporated in every generalized university or college course. Plato's belief in the value of mathematical thinking is still justified. In the years to come, Economics and Political Science, even Chemistry and Physics, will change so that our present ideas will seem absurd. But, as J. W. A. Young once remarked, in ten thousand years the square on the hypotenuse will still equal the sum of the squares on the sides.

4. The demand for scientists. The production of an adequate number of scientists is a problem which is now of great concern to the United States Government, as well it may be. Recently a five-volume Report to the President on Science and Public Policy, by John R. Steelman, was released. The fourth volume is devoted to Manpower for Research, and contains a one hundred page report by the Cooperative Committee on Science and Mathematics Teaching. This Committee is now a standing committee of the American Association for the Advancement of Science, under the chairmanship of Professor Lark-Horovitz, Secretary of the A.A.A.S. Thirteen prominent scientific societies have representatives on this committee. Professor Raleigh Schorling is the representative of our Association.

This is neither the time nor the place to give a detailed account of this report, but it is to be hoped that eventually it will reach every member of this Association. The Government's aim is directed toward the creation of (1) a corps of effective research scientists, and (2) an organized group of discerning science educators. The portion of this report written by the Cooperative Committee is primarily concerned with the early identification of youth with special talent in science and mathematics and with the proper nurture of these gifted youth through school, college, and the university.

It is of course high time that the Government took some interest in solving a problem which was to a large extent of its own creation. The short-sighted attitude of the Army during the war toward higher education did more damage than even a five-volume report to the President can now repair. While Canada and Australia and England nurtured their scientists and future scientists, we sent our A. S. T. P. boys to the Rapido River and the Anzio Beachhead.

The shortage of young scientists is indeed alarming. While it is due in large part to the cessation of the graduate schools during the war, it is aggravated by the great demands for research scientists by the Government and by Industry.

The latter has suddenly discovered the science of statistics, for instance. If there are enough men to go around, this is all to the good. Scientists are getting better and better salaries, until now the best of them are earning almost as much as an average big-league baseball player. Of course when the depression comes both Industry and Government will willingly relinquish their scientists to the colleges.

While all of us are in favor of adequate salaries for scientists, we cannot close our eyes to the impact of this sellers' market upon the colleges. A large corporation may employ a handful of scientists, whose salaries are an unimportant item on its budget. A university with a faculty of hundreds feels this competition keenly, and in reality cannot meet it. Older men may elect to remain in their teaching positions in spite of offers of increased salaries from Government and Industry. The younger men cannot be expected to do this, and they are not doing it.

The direct result of this situation may be the deterioration of higher education in the United States. The universities cannot train first-rate scientists with second-rate staffs. The fountain of knowledge may be shut off at its source.

The Government seems at long last to have recognized this state of affairs in the Steelman Report, but it remains to be seen if anything effective will be done about it. One obvious immediate remedy would be to close down many of the overlapping projects now being carried on with lavish expenditure, but we may be sure that this will not be done.

5. The public schools and the training of scientists. As a long-range program, however, the Steelman Report seems excellent in most respects. It recognizes that all is not perfect with our elementary and high school systems, and considers at length the problem of the early identification of youth with special talent in science and mathematics, and with the proper methods of handling them so that this talent may be developed as quickly and completely as possible. As the report states, a year or two lost can never be regained, not even if the student turns out to be a top-flight scientist.

I feel sure that at this point the Steelman Report has put a finger squarely upon the greatest weakness of our high school system. In the matter of educating the average child to become a good citizen, the schools are doing a fine job. In training the gifted for leadership, the record is not so good.

The mass production of high school graduates has had some unexpected by-products. We believe so firmly in the equality of men under the law that we assume them to be equal emotionally and mentally. At first we tried to offer everyone a college preparatory education. Many failed and were frustrated, so this system was considered unsatisfactory. The only obvious way out was to water down the contents of the courses to the capacities of the least capable, and this has been one of the aspects of "progressive education" during the past couple of decades. Since the mountain could not be persuaded to go to Mahomet, Mahomet was obliged to go to the mountain. This method may

produce citizens but it does not produce scholars.

For many high school pupils, the education which they receive is quite satisfactory. It is all they can absorb, they are not frustrated, and they are kept off the streets. They learn citizenship, although it must be admitted that their courses on citizenship do not seem to prevent them from littering the streets around the school with paper, much as they did before the Social Sciences were introduced into the high school curriculum. They are indoctrinated in the "American Way of Life," which means a mild contempt for scholarship and the belief that football is the only worthy masculine activity. With the substitution of swordsmanship for football, we have the way of life of the Norman nobles of the Middle Ages.

It is quite possible that some of these lads would have been frustrated by the old fashioned grammar school where grammar was taught instead of Social Studies. They will, eventually, be frustrated by life, but they will not take it too hard. Extraverts are not easily frustrated.

The greatest failure of our secondary schools is in their unsympathetic handling of the potential scholars. These superior students will be the future leaders of America, and their education is of primary importance. Without them, our civilization will fall. But they are frequently sensitive creatures, slow to mature, and possibly somewhat unlovely in adolescence. They are frequently unsympathetically handled by extravert teachers, and many of them are definitely frustrated. All they ask is to be allowed to pursue their serious interests at their own normal speed, but this is denied them, and they are made to feel that this desire is in some way abnormal. Frequently they find no kindred spirit even among their teachers.

European schools, and the American schools of forty years ago, cultivated more ground and ploughed it deeper than most of our modern schools. They had a curriculum of solid material so that the pupil had to develop at an early age the habit of serious study. In the grades he really learned spelling and grammar and arithmetic, and with a feeling of accomplishment following effort, came a genuine liking for the subjects. In high school he learned two languages, he learned to read them and to write them, and he learned their grammar. Possibly he learned to speak them. He also learned a little English, Ancient and American History, and a dab or two of the sciences. He took Algebra and Geometry as a matter of course, and frequently he took four years of mathematics. This was a college preparatory course. A student unable to complete this course went as far as he could with it. Modern experts say that this was bad, that his entire curriculum should have been changed and watered down. I stoutly maintain that this judgment is still unproved.

The colleges of today are facing an almost impossible task in trying to make scholars in four years out of unprepared students. What should be a leisurely course in College Algebra has become a mad drive to teach high school algebra to students who have already passed it at 97% in high school. Of course, the casualties are high but the colleges are trying to hold the line. But unless

relief comes soon, they will yield, and will spend two years doing what the high schools should have done.

What is the basic cause of this condition, and what is the remedy? I can do no better than to quote Dr. John Guy Fowlkes, Dean of the School of Education of the University of Wisconsin, from an address to his faculty last month: "More than ever before in the history of our country it is clear that teachers in our public schools need to be scholars."

It was an evil day when the direction of policy in our schools passed out of the hands of the scholars. When the direction of our medical schools shall pass from the hands of our doctors of medicine, and the direction of our law schools from the hands of our lawyers, they too will become ineffective.

This state of affairs is clearly due to the great increase in size which our school systems have undergone in recent years. There are not enough scholars to go around, and there is not enough money available in some local units to pay teachers adequately. This is the crux of the whole difficulty. High school teaching as a profession must be made more attractive to persons of both sexes who have a liking for the scholarly life.

The obvious answer to this is to increase taxes. We all have seen statistics comparing what we pay for cosmetics, tobacco, and liquor with what we pay for education. But even without any further increase in total expenditures, many schools could, under the administration of scholars, become more effective. Too much money is now spent on vocational guidance, bands, plays, card index systems, business machines and shop equipment, to say nothing of gymnasia and athletic paraphernalia. Now that I have burned my bridges behind me and admitted being a reactionary, I might as well state the belief that the damage done by basketball and track in the way of enlarged hearts far outweighs the benefits to the participants. Death results usually at about the age of fifty.

The high schools contain many capable and devoted scholars. But they are now in the minority, either numerically or politically, and have pretty generally lost control of the policy-forming bodies. Scholarship is not now the high ideal of the schools. It has been replaced by "Americanism," which is a rather vague concept somewhat beyond my powers of analysis. Probably it is "Wisconsinism" in Wisconsin, and "New Yorkism" in New York. It looks to me startlingly like "provincialism" everywhere.

The scholar is a reasonable person. He sees both sides of every problem and is quite willing to grant the other fellow every courtesy in an argument. Being very realistic, he is not positive that he is always right in every argument, and is a bit cautious in his declarations and claims. Moreover, he generally does not like an altercation. The scholar, therefore, is frequently an easy opponent for the extravert. The extravert knows that he is always right, because he can make more noise than his opponent. His beliefs have a basis in self-interest, but the extravert honestly does not realize this, and he becomes quite emotional about anyone who opposes him. He knows so definitely what he wants, and is so contemptuous of opposition, that he is impressive. Usually he gets his way.

As the schools of this country became "big business," their control passed from the hands of the scholars. The buildings became larger, the bus rides became longer, the frills began to multiply. Athletics and bands became more important than studies. Typewriting, shorthand, shop work, millinery, and home economics began to drive out the scholarly courses. Every teacher desired to become a principal so that he would not have to teach any more but could surround himself with secretaries, file clerks, vocational guidance directors, and assistant principals. To quote the title of an article in the Wisconsin Journal of Education last month, "The principal is an important person." Nothing was said of the importance of the classroom teacher.

Within the great system of our public schools there are many pressure groups working for the furtherance of their own best interests. I do not refer to anything so crass as the teachers who struck last winter and tied up many schools—it seems to be the American way. But there are groups who are perpetually striving for the over-emphasis of their own particular subjects or side shows at the expense of scholarly subjects. They do not hesitate to demand, for instance, that mathematics be eliminated to make room for political indoctrination.

As might be expected, the proponents of physical education and compulsory athletics are the most noisy and unreasonable, for they are the farthest from the scholarly type. At present they are pushing a campaign in Illinois to require athletics in all schools, from the first grade up. In Iowa, the only required subjects in high school are American history and government, and physical education. As if the farm boys in Illinois and Iowa did not have enough fatiguing work, and as if Nature did not keep every small boy in perpetual motion! Large sums going to manufacturers of athletic equipment might better be spent for the salaries of teachers.

There was a time when the colleges exercised some control over the secondary schools by demanding a measure of accomplishment from entering students, *and let no one persuade you that the colleges have no right to make such demands.* Those students who were headed for college had to have adequate instruction. This frustrated the pressure groups in the schools, and became the object of a concerted and violent attack. Most legislatures were persuaded to force the state-supported colleges to allow the high school principals to decide who is prepared for college and who is not.

This seems to be the critical point at which the scholars completely lost control of the schools. Long before this they had ceased to be in the majority in their own schools, but they had support from the colleges, most of which still remain scholarly institutions. But with the loss of this last restraining influence, degeneration has followed rapidly.

6. Obligations of the Association. The Mathematical Association of America was founded in 1916 with the avowed purpose of assisting in promoting the interests of mathematics in America. During the thirty-one years of its existence

it has been influential and respected. The Young Report on the reorganization of mathematics in the secondary schools was and is respected, and there have been other projects to our credit. But we must look forward rather than backward.

The Association must not be just another pressure group whose only purpose is to further the interests of its own members, and I don't think it has ever been such. Mathematics can hold its own in any fair competition. We must, however, work for a sane and realistic program of education, a return to the scholarly ideal.

With the great American experiment of mass education we should have no quarrel. But this project is being quite adequately handled, and presents no critical problem. The adequate education of scholars by scholars is not being well handled and needs our best thoughts. Call it the double-track system or what you will, there must be some way to give a genuine education to those who want it, and to persuade all those who are competent to want it. The ideal of scholarship must be made more honorable and more attractive.

Working alone, the Association can accomplish little. If we make common cause with other scholarly and scientific groups, we can be effective. The legislatures and state boards are not as a rule vicious or unfriendly, but they are not composed of educational experts and are subject to influence by pressure groups. One may believe that they frequently welcome our advice and are glad to support us if they have any evidence at all that they are doing the will of the people thereby. It is up to us to supply this evidence.

Just to show that a little effort may produce results, I want to tell you of recent happenings in Ohio. About three years ago when Dean Brandeberry of the University of Toledo became Chairman of the Ohio Section of the Association, he appointed a committee to see what, if anything, could be done about the low state to which mathematics in the state had fallen. No mathematics was required for graduation from high school, no mathematics was required for admission to many teacher-training institutions, and little if any was taught to the future teachers in these institutions. Under the chairmanship of Professor F. B. Wiley of Denison University, and later of Professor H. P. Fawcett of the College of Education of Ohio State University, this Association committee drew up resolutions and detailed programs of study and submitted them to the State Department of Education in Columbus. Within the present month the committee has learned that in the future one year of high school mathematics will be required for graduation. This is not enough of course, but it is a great deal.

There is some evidence that the anti-scholastic movement in Education, at least as it relates to mathematics, is on the wane in most places, and that the pendulum is about ready to swing back. Sooner or later this must take place, for if it were nothing more, mathematics is the back-bone of science and engineering. A little missionary work on our part at this time might prove surprisingly effective.

It seems clear that most of the opposition to mathematics in the schools

has been because of the idea that it stood in the way of mass education. I think we have no quarrel with mass education provided it does not interfere with the training of scholars. Instead of trying to force a high mathematics requirement on all schools, I think we should demand instead that those students who are going to go to college shall be adequately trained, and that the colleges shall have the right to decide whether a student is fitted to enter. If the poor student has a right to an education which will not frustrate him, by the same precept the talented student has a right to an education which will develop his talents as quickly and completely as possible. If the Steelman report is adequately implemented by Congress, steps will be taken to see that such an education is available.

With such an education available, the vocational guidance directors will have to be fought off so that all capable students will take this program. My daughter, who is now taking a college preparatory course in high school, has with considerable effort withstood the pressure of a vocational guidance director who insists that her course is not well balanced and that she should take home economics.

The matter of adequate training for teachers of mathematics is now a critical problem. In some Teachers Colleges practically none at all is required. At the University of Wisconsin, 15 semester hours beyond the calculus is necessary for the University Teachers Certificate. A joint committee of the Association and the National Council of Teachers of Mathematics is now considering the problem of training for teachers, and will soon have some suggestions to offer.

But experience in Wisconsin proves that a high paper requirement is not enough to ensure adequately trained and scholarly teachers. The number of students working for a Teachers Certificate with mathematics as a major has dropped everywhere in recent years. Thus the University of Michigan with 20,500 students has 5 who are looking forward to teaching mathematics in high school. There has not been a single student teacher in physics at Michigan during the last five semesters.

While the official requirements for high school teachers are high in these states, the actual teaching is being done by teachers with temporary permits who may have no qualifications at all. The only alternative would be to close the schools. It is rather futile, then, to talk of raising the price of tickets when the multitude crawls in under the tent.

7. Conclusion. All of these considerations lead us back to our original text: *Scholars are important persons.* They are more important than principals, than athletic coaches, than vocational guidance directors. Scholars are worthy of public respect, and should occupy a respected social position in this country as they do in most other countries. They should be given adequate salaries so that they can live as well as the physicians and lawyers, whose equals in public esteem they should be. The schools should be under the direction and management of the scholars, and the growing gap in both prestige and salary between

the principal and the teacher should be closed.

If such a conversion of the American people could take place, our school problems, and most of our political, economic, and social problems as well, would be solved. It is of course an idle dream. But scholars have always been dreamers, for only their dreams have made life endurable for them. And only those small bits of their dreams which have come to pass have made life endurable for the rest of the world.

RECENT PROGRESS IN COMPRESSIBLE FLUID THEORY

RUFUS ISAACS, North American Aviation, Inc.

1. Introduction. Modern technical developments have made the need for a workable theory of compressible fluids imperative. The classical techniques that have at times yielded successful solutions to similar applied problems here fall way short of the requirements. Let us survey this situation and analyze the nature of the difficulties which cause it.

We confine our attention to two-dimensional irrotational flows of non-viscous fluids. If the assumption of incompressibility is added, the classical development leads rapidly to a harmonious theory whose salient points are the following:

A. The stream and potential functions satisfy differential equations for which the boundary value problem admits of a unique solution.

B. These differential equations are linear and homogeneous. Thus linear combinations of solutions give further solutions. Usually there exist basic sequences of solutions so that an arbitrary solution may be formed from the limits of linear combinations of them.

C. Each differential equation is a Laplace equation. Thus solutions may be put into correspondence with analytic functions of a complex variable by the operation:

"Take the imaginary part." (1)

Aspect **A** continues to hold for a larger class of problems. In fact, for any problem amenable to differential equation treatment, mathematical uniqueness should follow from physical uniqueness if the mathematical problem is correctly formulated from the (necessarily idealized) physical situation. Aspects **B** and **C** each lead us into well-studied mathematical domains, and it is not surprising that incompressible fluid theory should be rich in effective, and often elegant, results.

But even with such advantages, all the troubles of the applied mathematician are not dispelled. The boundary value problem in practice consists in being given certain numerical data and being required to supply other such data with

a (sometimes tacitly) prescribed degree of accuracy. The only means possessed by men to make the transition are the common arithmetic operations on integers. Thus the *method* is reduced to finding a program of such operations.

This can often be done. For example, we may dot the domain of the sought function with a lattice of points and replace the function by a set of numbers, one at each point. The differential equation is replaced by a suitable difference equation. The problem is replaced (usually) by an algebraic one where we have an unknown for each lattice point. If the original differential equations are linear, so will be the algebraic ones for the unknowns. Then we possess the required program of arithmetic operations.

The drawback to such methods is that the number of operations may be enormous; so much so, in fact, that the time and labor required may exceed the economic value of the solution. With such advantages as **B** and **C** present, the amount of labor can sometimes be reduced into the practical range by utilizing existing known properties and tabulations of functions. But even in cases where the pure mathematician has supplied a sophisticated approach with impeccable convergence proofs, the numerical calculation may still be prodigious.*

The most promising remedy, in the author's opinion, lies in the use of modern computing devices such as punch card machines, the Automatic Sequence Controlled Calculator at Harvard University, or Aniac, the Bell Telephone Computer. The principle of these machines is to compound arithmetic calculators so as to carry through a program of operations automatically; that is, the result of one operation can appear in the data of the next. Thus vast new possibilities in applied mathematics are opened up.

For non-linear problems there may be a still more fundamental difficulty. Even if the resources are at hand for carrying out a vast program of arithmetic operations, there is still the primary problem of formulating the program. The usual devices, some of which we have mentioned, no longer suffice. In fact, sometimes even the general qualitative nature of the solution may be obscure. Thus there are two stages in the difficulties of compressible fluid theory; only recently has effective progress been made in overcoming them. In the ensuing paragraphs, we outline the main steps.

2. The main steps. First we recall some terms. The *potential* is that function whose gradient is the velocity of flow. It exists when the flow is *irrotational*. Descriptively, this term means that small regions of fluid do not rotate; precisely it means the vanishing of a certain line integral for all closed curves. The dynamical justification of irrotationality can be found in the classic researches of Helmholtz. The chief property of the *stream function* is that it is constant along the stream lines; hence its knowledge leads to a picture of the flow pattern.

* For example, let the skeptic attempt the alternating procedure of Schwarz with two slowly convergent infinite series. See Courant-Hilbert, Methoden der Mathematischen Physik, T. II, p. 264.

For an incompressible fluid the stream function ψ and the potential function ϕ each satisfy Laplace's equation. In fact $\phi+i\psi$ is a monogenic function of $x-iy$, where x, y are the cartesian coördinates in the plane of the flow.

For the incompressible case we must assume an equation of state, which we take to be

$$p = A + \sigma\rho^k \tag{2}$$

where p is the pressure, ρ the density, and A, σ, and k are constants. The equations satisfied by ϕ and ψ are

$$a^2[\phi_{xx} + \phi_{yy}] = \phi_x^2\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + \phi_y^2\phi_{yy}, \tag{3}$$

$$\psi_{xx}\left[1 - \frac{1}{(\rho_0 a_0)^2}\left(\frac{\rho_0}{\rho}\right)^{k+1}\psi_y^2\right] + \psi_{yy}\left[1 - \frac{1}{(\rho_0 a_0)^2}\left(\frac{\rho_0}{\rho}\right)^{k+1}\psi_x^2\right] + \frac{2}{(\rho_0 a_0)^2}\left(\frac{\rho_0}{\rho}\right)^{k+1}\psi_x\psi_y\psi_{xy} = 0, \tag{4}$$

together with

$$a = [a_0^2 - \tfrac{1}{2}(k-1)(\phi_x^2 + \phi_y^2)]^{1/2}, \tag{5}$$

$$\left(\frac{\rho_0}{\rho}\right)^{1-k} = 1 - \frac{k-1}{2(\rho_0 a_0)^2}\left(\frac{\rho_0}{\rho}\right)^2[\psi_x^2 + \psi_y^2]. \tag{6}$$

Here a is the local velocity of sound; a_0 and ρ_0 mean a and ρ at zero velocity. It is apparent that these equations are not only non-linear, but complicated. However, the conveniences of aspects **B** and **C** may be restored by means of the so-called hodograph plane. This plane is one in which the horizontal and vertical velocity components u and v are the coördinates instead of x and y. The functions ϕ and ψ may be regarded as functions of u and v, and hence the stream and equipotential lines may be depicted in the hodograph plane. This gives rise to a flow pattern (the hodograph), generally quite different in appearance from that in the physical plane (*i.e.*, the x, y plane). For the incompressible case, ϕ and ψ are harmonic in the hodograph plane and so it appears to offer little advantage. But, as Chaplygin [1] has shown, for compressible fluids the equations become linear in the hodograph plane.

Bergman [3, 5, 7] modifies this idea (in the subsonic case) by using what he terms the pseudo-logarithmic plane instead of the hodograph plane. Here the coördinates are λ and θ where

$$\lambda = \tfrac{1}{2}\log\frac{1-T}{1+T} + \frac{1}{2h}\log\frac{1+hT}{1-hT}, \tag{7}$$

$$h = \left(\frac{k-1}{k+1}\right)^{1/2}, \qquad T = (1-M^2)^{1/2},$$

$$M \text{ (the Mach number)} = \frac{V}{a_0}\left[1 - \tfrac{1}{2}(k-1)\left(\frac{V}{a_0}\right)^2\right]^{1/2},$$

$$V \text{ (velocity of flow)} = (u^2 + v^2)^{1/2},$$

and θ is the angle which the flow velocity makes with the horizontal axis.

The equation for ψ now assumes the form

$$\psi_{\lambda\lambda} + \psi_{\theta\theta} + 4N\psi_\lambda = 0 \tag{8}$$

where

$$N = -\frac{k+1}{8}\frac{M^4}{(1-M^2)^{3/2}}.$$

Bergman has developed a theory of operators which define a correspondence between analytic functions of complex variables and the solutions of certain second order partial differential equations [2, 4]. Thus it is the analogue of the operator (1) for harmonic functions. Applying the appropriate operator to (8) yields the following result:

Let g be an arbitrary analytic function. Put

$$Z = \lambda - i\theta$$

$$g^{[0]}(Z) = g(Z), \qquad g^{[n]}(Z) = \int_0^Z g^{[n-1]}(z)dz.$$

Then

$$\psi(\lambda, \theta) = \Im_m\left\{(1-M^2)^{-1/4}(1 + \tfrac{1}{2}(k-1)M^2)^{-1/2(k-1)}\left[g(Z) + \sum_{n=1}^{\infty}\frac{(2n)!}{2^{2n}n!}Q^{(n)}(2\lambda)g^{[n]}(Z)\right]\right\} \tag{9}$$

is a solution of (8).

In (9) the sequence of functions $Q^{(n)}$ is defined as follows:

$$(2n+1)Q^{(n+1)}(\lambda) = -\frac{d}{d\lambda}Q^{(n)}(\lambda) - 4\int_{-\infty}^{\lambda}F(\alpha)Q^{(n)}(\alpha)d\alpha \quad (n = 0, 1, \cdots), \tag{10}$$

$$Q^{(0)}_{(\lambda)} = 1;$$

$$F(\lambda) = \frac{1-T^2}{16(1-h^2)^2T^6}\left[5 - (1+6h^2)T^2 - (5-4h^2)T^4 + (1+2h^2)T^6\right]. \tag{11}$$

The function F is fully defined by the conjunction of (7) and (11).

3. Remarks. Thus the problem is solved in the sense that methods of solution have been found. These methods bear an obvious analogy to the incom-

pressible case. The chief computational task needed to open the way toward numerical usage is the evaluation of the $Q^{(n)}$. But these can be tabulated once and for all; with their aid all subsonic (and with modifications, supersonic) flows can be found.

One way of handling the problem is to replace the derivative and integral expressions in (10) by suitable interpolative formulas. The computation can then be done by mechanical devices of the type described earlier. For details of the technique, see [8].

Once the $Q^{(n)}$ are available, we can construct the aggregate of flow patterns. Let us suppose ourselves faced with a particular problem, for example, the flow around a given obstacle. For the incompressible case we solve this problem by one of the known methods and thus find the appropriate analytic function g. Using the operator (9) we can obtain the flow in the pseudo-logarithmic plane. A routine procedure [6] enables us to pass to the physical plane. The result is a flow about an obstacle slightly distorted from the given one. By making a suitable counterdistortion at the outset it is possible to obtain the true pattern.

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THE TEACHING FELLOW PROGRAM AT MICHIGAN

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The employment of part-time teaching assistants or fellows is now a widespread and necessary practice. It should be continued beyond this period of staff shortages for the sake of the teacher-training of future instructors and professors as well as for the sake of financial aid to struggling scholars.

In the October, 1946, issue of the MONTHLY B. W. Jones [1] summarized the replies to an inquiry which he addressed to several colleges and universities concerning their supervision of teaching assistants. At that time a revised and expanded program for the supervision and training of the teaching fellows in mathematics was being instituted at the University of Michigan. It is hoped that an account of this program at the beginning of its second year may be of interest, and may draw forth exchanges of information about similar programs.

The class of fellowships entitled "teaching fellowships" was established at the University of Michigan in 1932 to be filled by graduate students who "exhibit both research capacity and teaching ability." It was stated at that time that "the Graduate School assumes a supervisory function with respect to the inservice teacher-training which each department by its appointment of teaching fellows obligates itself to perform" [2]. With the sudden growth of the Mathematics Department's teaching fellow group to thirty-five in 1946–47 and forty-five in 1947–48 it became necessary to revise and expand the previously existing teacher-training program. The objectives of this program were considered to be: (1) to provide for the welfare of freshmen and sophomore students by maintaining a high quality of instruction, (2) to serve thereby the best interests of the University as an educational institution, (3) to help the teaching fellow, frequently inexperienced and sometimes lacking in confidence, by making his present job easier, and by providing training in the work to which many of his later years will probably be devoted.

The program for this year consists of three major parts, namely: (1) regular group lecture and discussion meetings, (2) a "consultation service," (3) classroom "visitations" followed by individual conferences. This is in general the same as last year's program, modified, however, in many details as the result of the suggestions, and vigorous, interested, and cooperative criticisms of last year's teaching fellows.

The group meetings are of two types, weekly meetings of the entire group and less frequent meetings of those teaching a particular course. At the weekly meetings of the entire group such topics as the following, some taking more and some less than one meeting, will be discussed: (a) administrative details of course schedules, texts, attendance rules and grading at this University, (b) general classroom teaching hints and advice, (c) different methods of teaching (*e.g.*, lecture, heuristic, and board work or laboratory methods), (d) suggestions for planning a class hour (*e.g.*, read the text, seek to show a motivation for your discussion, select examples, try to provide some variety in your procedure within

each class hour and from day to day), (e) typical student errors and teaching trouble spots to be anticipated, (f) the nature, sources, and use of enrichment materials (*e.g.*, historical ideas, models, applications or connections with other fields), (g) how to prepare, give and correct examinations. The word "discussed" was used advisedly in connection with these meetings because the group is encouraged to present questions, points of disagreement, or alternative procedures at any time. Actually our problem here, at the outset, was to limit discussions and digressions, not to encourage them. However, democratic procedures and discussions, properly controlled, increase interest and bring out profitable variations in viewpoint. After the topics of most significance for classroom application have been covered, the general meetings will be held less frequently and will consist of talks by faculty members, movies and slide films, criticisms of texts and the curriculum, and consideration of the literature of collegiate teaching.

The small groups, each of which includes those teaching a particular subject, consider the schedule for that course, what topics, if any, are optional, specific errors and teaching problems encountered in that course, the motivation and development of particular topics, devices and materials that may be used in the next few weeks of classes. Each of these groups meets once every four weeks.

The consultation service is represented by the posting of a rather complete schedule of office hours by the sponsor or supervisor. (He has no official title, but from the resemblance in function and name to a popular comic strip character, he has been termed "Available.") During these hours he checks, before they are administered, the first two or three examinations given by the new fellows, discusses with them the observations made upon visits to their classes, and more than this, tries to answer, as they are brought in, all types of questions ranging from "When and where do we get paid?" to "Why does the author of this text present his ideas this way?," and "What should one do for (or to) students who cheat, are absent unduly, are misclassified?"

As the third major part of this program, the supervisor attempts to visit, at least twice, each time for a full hour, the class of each teaching fellow, and to discuss his observations with the fellow soon after the visit. A more ambitious visiting program might be desirable but seems impossible now in terms of the time available. The entire program is administered by one person. An allowance is made for the time required to supervise the teaching fellows in determining his class load.

A minor feature of the program has been the grouping together of the teaching fellows' offices and the stocking of a nearby cupboard with old and new standard textbooks, teacher's handbooks, such as J. W. Cell, *Engineering Problems Illustrating Mathematics*, R. C. Yates, *Curves*, historical works by D. E. Smith, Florian Cajori, and R. C. Archibald, works related to teaching, such as Seidlin, *A Critical Study of the Teaching of Elementary College Mathematics*, Felix Klein, *Elementary Mathematics from the Advanced Standpoint*,

Pólya, *How to Solve It*, copies of pertinent articles from the MONTHLY, such as Campbell, "Advice to the Graduate Assistant" [3], "survey" texts such as those by Dresden, Cooley, et al., Merriman, Richardson, and finally some of the thought-provoking new texts such as Murnaghan's *Analytic Geometry* and Albert's *College Algebra*.

To evaluate the success of this program is difficult. One can not measure effectively how good the teaching has been, let alone know how much worse it might have been without such a program. The replies found on questionnaires filled out anonymously by the teaching fellows at the end of each semester indicated that by a large majority (but not unanimously) they felt all three phases of the program were helpful and worthwhile. The general meetings were considered least valuable. We are trying to correct those features of them which drew the most criticism. The general feeling with regard to "visitations" was that though a little unpleasant they were helpful and an especially good idea for the other teaching fellows. It is of interest that on several occasions people have asked for more visits. The supervisor feels that the program has at least resulted in an interest in and thought for teaching and curricular problems which, combined with the recognition of teaching as a real job in itself, have produced a desirable high morale among the teaching fellow group.

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THE SHORT-CUT PROBLEM

CHANDLER DAVIS, Harvard University

1. Introduction. The following problem in plane geometry concerns the shortest path from one point to another along a network of streets. To make the problem definite, let the streets be a family of straight lines, arbitrary in position and direction, infinite in length, and finite in number.

We will use the following notation. Let axes be chosen so the starting point O is at $(-1, 0)$ and the destination O' at $(1, 0)$; it is of course assumed that at least one line passes through each of these points. We will discuss properties of paths from O to O', composed entirely of segments of the given lines. Let the angle from the positive x direction to the direction of a line a be designated by $\theta(a)$, where the function θ has values in the range $-\pi < \theta \leqq \pi$. If we are considering using a segment of a as part of a path from O to O', we will understand $\theta(a)$

to have the value corresponding to that sense in which we consider traversing a. It may happen that we will want to give the angle of a, not with respect to the x direction, but with respect to the direction AB; in this case we write $\theta_{AB}(a)$. Thus, for example, $\theta(a) \equiv \theta(AB) + \theta_{AB}(a) \bmod 2\pi$.

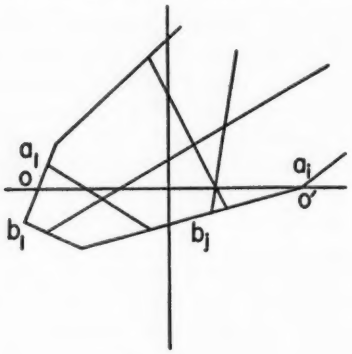

Fig. 1

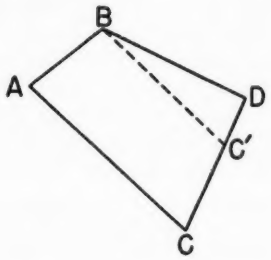

Fig. 2

Consider a line from the given family such that O and O' do not lie on opposite sides of it. Clearly the shortest path does not cross it, and we may for most purposes eliminate all parts of the plane on one side of it. After we have eliminated such irrelevant regions, we are left with a region like that in Figure 1, that is, a convex polygon, which however need not be closed. Call the segments of this polygon above the x-axis, $a_1, a_2, \cdots, a_i$, taken in order from O to O'; similarly call the segments below the x-axis $b_1, b_2, \cdots, b_j$. We have immediately

$$\pi > \theta(a_1) > \theta(a_2) > \cdots > \theta(a_i) > -\pi$$

$$-\pi < \theta(b_1) < \theta(b_2) < \cdots < \theta(b_j) < \pi$$

$$\theta(a_1) \geqq 0, \qquad \theta(a_i) \leqq 0$$

$$\theta(b_1) \leqq 0, \qquad \theta(b_j) \geqq 0$$

$$\theta(a_1) - \theta(b_1) \leqq \pi$$

$$-\theta(a_i) + \theta(b_j) \leqq \pi.$$

Those lines which are not of either type a or b we will call l-lines. In general the shortest path will include l-lines as well as a and b-lines.

A line m will be called *reflex* if $\theta(m) > \theta(a_1)$ and $\theta(m) > \theta(b_j)$, *or* if $\theta(m) < \theta(b_1)$ and $\theta(m) < \theta(a_i)$. Otherwise m will be called *direct*. Note that all "boundary" lines (types a and b) are direct.

The theorem to be proved can now be stated in the following form.

THEOREM. *The shortest path from O to O' includes no reflex segments.**

* This theorem, more general than the author's earlier result, is due to J. T. Tate, Jr., of Princeton.

2. Two lemmas.

LEMMA 1. *Suppose we have two alternative two-segment paths ABD and ACD from point A to point D, with* $\theta_{AD}(AB)>0$, $\theta_{AD}(AC)<0$. *Suppose further that* $\theta_{AD}(AB)<\theta_{AD}(CD)$. *Then if ACD is shorter than ABD we must have* $\theta_{AD}(AC)>\theta_{AD}(BD)$.

For, suppose instead $\theta_{AD}(AC)\leqq\theta_{AD}(BD)$. (See Figure 2.) Take the line through B parallel to AC; it meets CD in C', which lies between C and D (or coincides with D). Then comparing the lengths of the paths, we have $ACD>ABC'D\geqq ABD$, which proves the Lemma.

This Lemma and its various equivalent forms (for example, that in which $\theta_{AD}(AC)$ is taken less than $\theta_{AD}(BD)$, and $\theta_{AD}(CD)<\theta_{AD}(AB)$ is proved) will be used frequently in what follows.

LEMMA 2. *The first l-segment of a shortest path from O to O′ cannot be reflex.*

Suppose without loss of generality that the path leaves O along the upper boundary segments $a_1, a_2, \cdots, a_\mu$. Then let it leave a_μ at A along the reflex segment AP_1, so that $\theta(AP_1)<\theta(b_1)<0$, $\theta(AP_1)<\theta(a_i)$. From P_1 it proceeds along l-lines through points $P_2, P_3, \cdots, P_r, B$, where B lies on the boundary. Then, $\theta(P_1P_2)>0$, $\theta(P_2P_3)<0, \cdots$. (See Figure 3.) For, if the lines do not alternate in this way between positive and negative θ, then obviously one of them is crossed more than once by the path in question.

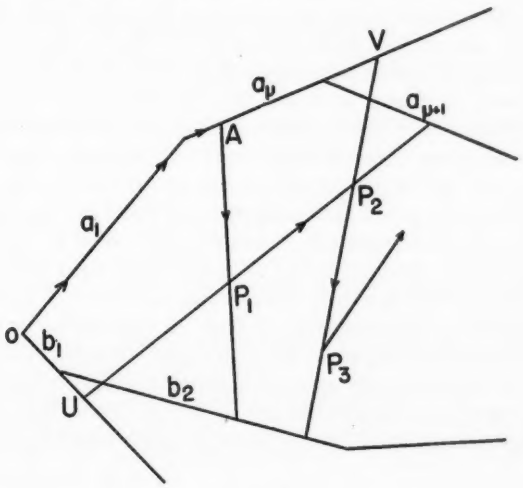

Fig. 3

Now let P_1P_2 meet b_1 at U, and imagine a line drawn from O to A. The path OAP_1 is shorter than the actual path $Oa_1a_2\cdots a_\mu AP_1$, hence by hypothesis

shorter than OUP_1; also $\theta_{OP_1}(OU) > \theta_{OP_1}(AP_1)$. Hence Lemma 1, when applied to quadrilateral OAP_1U, gives $\theta_{OP_1}(P_1P_2) > \theta_{OP_1}(OA) > \theta_{OP_1}(a_\mu)$; $\theta(P_1P_2) > \theta(a_\mu)$.

Similarly, we take the intersection V of P_2P_3 with a_μ and apply Lemma 1 to quadrilateral AVP_2P_1 to obtain $\theta(P_2P_3) < \theta(AP_1)$. Successive applications of Lemma 1 give $\theta(P_3P_4) > \theta(P_1P_2)$, $\theta(P_4P_5) < \theta(P_2P_3)$, $\theta(P_5P_6) > \theta(P_3P_4)$, $\cdots$.

Take first the case where ν is odd; B must lie on an a-line, say a_k. Since $P_{\nu-1}P_\nu$ is reflex from the inequalities derived above, and a_k is direct, $\theta(P_{\nu-1}P_\nu) < \theta(a_k)$; also we know that $\theta(P_\nu B) > \theta(P_{\nu-2}P_{\nu-1})$. Then the assumption that the chosen path gives the shortest distance from $P_{\nu-1}$ to B leads, by Lemma 1, to a contradiction.

If on the other hand ν is even, B lies on a b-line, b_λ. In the event that $\theta(P_{\nu-1}P_\nu) > \theta(b_\lambda)$ we prove a contradiction as in the preceding case; so take $\theta(P_{\nu-1}P_\nu) \leqq \theta(b_\lambda)$. $P_{\nu-1}P_\nu$ clearly meets a_i in some point W above the x-axis, and application of Lemma 1 to quadrilateral $P_\nu WO'B$, in which $\theta(P_\nu W) \leqq \theta(BO')$ and $\theta(P_\nu B) < \theta(WO')$, gives a contradiction. (That part of the chosen path between B and O' has been replaced by the construction of a single straight line.)

Lemma 2 is therefore proved; note that the cases $\nu < 3$, though apparently not covered, are actually obtained by the same method.

3. The main theorem. The theorem now follows immediately by induction on the number n of l-segments in the path in question. For if $n = 1$ then Lemma 2 is equivalent to the theorem. Also suppose the theorem true for all m which are less than or equal to $n-1$, and consider a path having n l-segments. The first segment AP_1 on which this path leaves a boundary (say the a-boundary) is, by Lemma 2, direct; thus $\theta(AP_1) \geqq \min\{\theta(b_1), \theta(a_i)\}$. But the portion of the path between A and O' must be a shortest path for those two points. Consider the short-cut problem related to A and O'; AP_1 will now be a boundary line, so there will be at most $(n-1)$ l-segments in this new problem. By the induction hypothesis, no segment is reflex with respect to AO'; but any segment reflex with respect to OO' would be so with respect to AO', so the theorem is proved.

It is clear that the proof has nowhere used the fact that the boundary is made up of straight-line segments; it could equally well be any convex curve, not necessarily closed, through O and O'. (All l-lines must still be straight.)

The following further problem has been suggested by J. T. Tate, Jr. Given some particular figure as boundary, to find for each point in the interior the maximum and minimum angles of lines through that point which can for some configuration of l-lines be portions of shortest paths. For some points these maximum and minimum conditions will be sharper than those obtained by the present theorem; indeed there can be points through which no shortest path can pass.

THE CHAUVENET PRIZE FOR MATHEMATICAL EXPOSITION

The Chauvenet Prize for Mathematical Exposition has been awarded to Professor P. R. Halmos of the Institute for Advanced Study for his paper entitled "The Foundations of Probability," published in this MONTHLY for November, 1944. This most recent award of the Prize "for a noteworthy expository paper published in English by a member of the Association" covers the three-year period, 1944–'46.

The Association first established the Chauvenet Prize in 1925. At that time it was specified that the award was to be made every five years for the best article of an expository character dealing with some mathematical topic, written by a member of the Association and published in English during the five calendar years preceding the award. The Prize was not to be awarded for books. Originally the amount of the award was fixed at one hundred dollars.

At a later date it was decided to award the Prize every three years, and the amount was changed to fifty dollars. In 1942, it was further specified that only such papers would be considered as "came within the range of profitable reading of Association members."

The publication program of the Association recognizes the importance of expository writing. The editors of the MONTHLY welcome short expository articles that treat mathematical concepts of modern importance. Longer manuscripts may be submitted to the Committee on the Slaught Memorial Papers or to the Editorial Committee on the Carus Monographs.

ERRATA, VOLUME 54

The following errata in Volume 54 have been called to the attention of the editors:

H. E. Salzer, *The approximation of numbers as sums of reciprocals.*

p. 135, at the bottom of the page, replace

$$\frac{1}{a_i^2 + a_i} \quad \text{by} \quad \frac{1}{a_i^2 - a_i}.$$

W. C. G. Fraser, *An inversion formula for an integral related to Dirichlet series.*

p. 586: The location of the author should be given as Rensselaer Polytechnic Institute instead of Rutgers University.

Kurt Godel, *What is Cantor's Continuum Problem?*

pp. 515–525: In the footnotes 1, 2, 20, 23 and in the first line of p. 521 the numbers of the footnotes quoted should read: 14, 15, 17, 20, 23, 26 instead of 12, 13, 14, 17, 19, 21, respectively.

In footnote 29, *l.c.* 6 should be replaced by *l.c.*[6].

In the second line of p. 524 Π_i should be replaced by $\prod_i$.

In the fifth line of p. 525 the word "at" should be replaced by "are."

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

ROTATION OF THE TANGENT TO A HYPOCYCLOID

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A surprising number of the properties of hypocycloids are related to the turning of the tangent, as the point of contact R between the rolling circle r and the fixed circle f moves from the cusp A to the position determined by the parametric angle θ. If C is the center of r and X is the point describing the hypocycloid, then the tangent to the hypocycloid at X always passes through S, the point of r opposite R. Also, from the equality of arcs RX and RA of r and f respectively, we have angle XCR equal to $a\theta/b$, where a and b are radii of f and r respectively. Then angle XSR is $a\theta/2b$, which means that the tangent SX has turned from its initial direction OA through the angle $\tau=(2b-a)\theta/2b$. The motion of the supporting point S about its circle s, accompanied by the uniform rotation of the line SX, determines this family of lines whose envelope is seen to be some hypocycloid or epicycloid.

We may evaluate τ for a deltoid upon replacing a by $3b$, thus obtaining $-\theta/2$. To identify the envelope of the Simson line, we may take S on the nine-point circle. Then we know (N. A. Court, *College Geometry*, p. 116) that the Simson line rotates half as fast in the opposite sense as S moves about the nine-point center. It would be hard to imagine a simpler proof for the famous theorem that the envelope of the Simson lines of a fixed triangle is a deltoid having the nine-point circle of the triangle as inscribed circle.

For the astroid, a equals $4b$, and τ becomes $-\theta$. Now S bisects the radius OR of f, and the tangent may be drawn joining the projections of R on the perpendicular cusp tangents. Then X is the projection of R on this tangent. The proof lies in noting that angle XSR is 2θ and angle XCR is 4θ. This not only furnishes a neat construction for the astroid but also proves very simply the textbook proposition concerning the constancy of the length of the segment intercepted by the axes on the tangent.

If $a=2b$, the hypocycloid has two cusps and τ vanishes, proving that the curve reduces to a diameter of f. If $a=b$, then $\tau=\theta/2$. Then S moves on f opposite R and the hypocycloid of one cusp reduces to the single point A. The general point of an epicycloid can be found by reflecting the point of the corresponding hypocycloid across the tangent at R. Thus the cardioid is the reflection of A across the variable tangent to f.

The deltoid has many interesting properties, some of which can be approached very directly from the present point of view. For instance, if the rolling curve r' has radius $b'=2a/3$, the point marked on r' and starting from A traces the same curve formed by letting $b=a/3$. To prove this, note that corresponding to this value of b', $\tau'=\theta/4$, and OS' is $-OR/3$. Then angle $PS'R$

is $3\theta/4$, where the moving point P is the projection of R on $S'P$. If S is such that OS is $OR/3$, it is the center of r' and angle PSR is $3\theta/2$, which shows that the arc RP of r' equals arc RA of f. That this point P coincides with a point of the deltoid with $b=a/3$ may be shown by drawing OT, where T is on f and angle AOT is $-\theta/2$. Let U be such that $OU=OT/3$. Since angle UOS is $3\theta/2$, $S'P$ passes through this same point U of the circle s on SS' as diameter. Thus angle TUP is $3\theta/4$, which implies that the line UP is a tangent to the deltoid generated by the circle on UT as diameter. Since UT and SR make numerically equal angles with $S'U$, the projections of R and T on this line coincide, which completes the proof. A result which is obvious at this point is the fact that the chord which f determines on the normal at P is trisected by P.

Another well known result which follows easily at this stage is the theorem that the segment determined on the tangent to one branch of the deltoid by the other two branches is constant. For angle PSR is $3\theta/2$, so that SP is tangent to one branch at X, where X is the projection of R on SP. When r' starts rolling with P at A, the point Q of r' opposite P lies on the deltoid and therefore stays on the same curve as r' rolls. Thus the segment PQ referred to in this proposition is a diameter of the circle r'. Also RP and RQ are perpendicular normals to the deltoid and $S'P$ and $S'Q$ are perpendicular tangents. We can accordingly announce the following results: Two of the normals from any point of f to the deltoid are orthogonal, and the feet of the three normals are collinear. Likewise, the intersection of orthogonal tangents to the deltoid describes the circle s (the orthoptic of the deltoid).

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania.

A GENERALIZATION OF THE RATIO TEST FOR SERIES

R. J. DUFFIN, Carnegie Institute of Technology

While teaching the chapter on series in the calculus, it occurred to the writer that the ratio test and the alternating series test could formally be combined into a single test. Thus, according to the ratio test, a series $a_1+a_2+\cdots$ (whose terms have like signs) is convergent if $0\leqq a_{n+1}/a_n\leqq r$ where $0<r<1$. According to the alternating series test, a series is convergent if $a_n\to 0$ and $-1\leqq a_{n+1}/a_n\leqq 0$. Is it possible, then, to combine these two tests into a single test which is valid regardless of the signs of the terms a_i, and which is more general than the ratio test for general series which states that convergence follows if $0\leqq |a_{n+1}/a_n| \leqq r$ where $0<r<1$? An affirmative answer is given by the following result:

THEOREM 1. *A series* $a_1+a_2+\cdots$ *of real numbers is convergent if* $a_n\to 0$ *and if the ratios* a_{n+1}/a_n *satisfy the inequality* $-1\leqq a_{n+1}/a_n\leqq r$ *where* $0<r<1$.

The following proof of this theorem should be comprehensible to many elementary students. Denote the ratio a_{j+1}/a_j by z, so $-1\leqq z\leqq r<1$. If z is a negative number, then $1+z=1-|z|$. Furthermore $1-r\leqq 1+r$. Forming the product of these relations between positive numbers gives

$$(1+z)(1-r)\leqq(1-|z|)(1+r). \tag{1}$$

The same inequality holds if z is positive or zero because in this case $1+z\leqq 1+r$ and $1-r\leqq 1-|z|$. Denoting $(1+r)/(1-r)$ by k we can express (1) in the form

$$|1+a_{j+1}/a_j|\leqq k(1-|a_{j+1}/a_j|).$$

Multiplying through by $|a_j|$

$$|a_j+a_{j+1}|\leqq k(|a_j|-|a_{j+1}|). \tag{2}$$

Let $R_{mn}=a_m+a_{m+1}+\cdots+a_{n-1}+a_n$. Then

$$\begin{aligned}
2R_{mn} &= a_m+(a_m+a_{m+1})+\cdots+(a_{n-1}+a_n)+a_n;\quad\text{or}\\
2|R_{mn}| &\leqq |a_m|+|a_m+a_{m+1}|+\cdots+|a_{n-1}+a_n|+|a_n|\\
&\leqq |a_m|+k(|a_m|-|a_{m+1}|)+\cdots+k(|a_{n-1}|-|a_n|)+|a_n|\\
&= (1+k)|a_m|+(1-k)|a_n|\\
&\leqq (1+k)|a_m|.
\end{aligned}$$

By hypothesis $a_m\to 0$ as $m\to\infty$, so $R_{mn}\to 0$ independent of n. Thus by the Cauchy-sequence criterion for convergence it follows that the series converges.

If S is the sum of the series then the above inequality gives $|S|\leqq|a_1|/(1-r)$. Note that this inequality includes the well known inequalities for geometric series and alternating series.

We now give a generalization which allows the a_n to be complex numbers.

THEOREM 2. *A series* $a_1+a_2+\cdots$ *of complex numbers is convergent if* $a_n\to 0$ *and if the ratios* a_{n+1}/a_n *lie in or on a polygon which is contained in the closed unit circle, and which touches the circle at one and only one point* w, *and* $w\neq 1$.

Proof. Again denoting the ratio by z we see that $1-|z|$ is the shortest distance from z to the unit circle and that $|w-z|$ is the distance from z to the point w on the unit circle. It is geometrically obvious that there is a constant $k>1$ (dependent on the shape of the polygon) such that $|w-z|\leqq k(1-|z|)$. Thus

$$|wa_j-a_{j+1}|\leqq k(|a_j|-|a_{j+1}|). \tag{3}$$

But $(w-1)R_{mn}=-a_m+\sum_m^{n-1}(wa_j-a_{j+1})+wa_n$.

$$|w-1||R_{mn}|\leqq|a_m|+k\sum_m^{n-1}(|a_j|-|a_{j+1}|)+|a_n|\leqq(1+k)|a_m|.$$

This completes the proof.

It is of some interest to obtain an inequality for the sum of the series in terms of the shape of the polygon. The relation $|w-z| = k(1-|z|)$ for $k>1$ defines a closed curve L of points z. If $z=x+iy$ and $w=a+ib$ the above relation becomes $\{(x-a)^2+(y-b)^2\}^{1/2}=k-k(x^2+y^2)^{1/2}$. The number of points of intersection of L with the straight line $y=c$ is the same as the number of points of intersection of the curve $y_1=\{(x-a)^2+d^2\}^{1/2}$ and the curve $y_2=k-k(x^2+c^2)^{1/2}$ where $d=b-c$. If neither d nor c are zero y_1 is concave upward and y_2 is concave downward so these curves may intersect in at most two points. If $d=0$ then y_1 is an angle function opening upward so if $c\neq 0$ the curve y_2 could intersect y_1 at most twice. If c is also zero then y_2 is an angle function opening downward but with slopes differing from that of y_1, so y_2 and y_1 could intersect at most twice. The remaining case is symmetrical. Thus L can intersect a line parallel to the x-axis in at most two points. Using a rotated coördinate system does not change the form of the equation of L but only the values of a and b. Thus L can intersect any straight line in at most two points, in other words L is a convex curve.

Let g_j be the distance of the jth vertex of the polygon to the circle and let h_j be the distance to the point w. Let $k=\max h_j/g_j=h/g$ (excluding the vertex at w). Clearly the points z' of the interior of L satisfy the relation $|w-z'| < k(1-|z'|)$ so the vertices of the polygon cannot lie outside L. Moreover, since L is convex the sides of the polygon must lie inside L. Thus we may use this value of k in the proof of Theorem 2 and obtain

THEOREM 3. *If S is the sum of the series $a_1+a_2+\cdots$, then it follows that* $|S| \leqq (1+h/g)|a_1|/|w-1|$.

Two suggested extensions of Theorem 2 are false. It is not permissible to have more than one vertex of the polygon touch the circle. It is not permissible to have the ratios lie within a circle inside but tangent to the unit circle.

The proof of the following theorem parallels that of Theorem 2.

THEOREM 4. *A series $a_1+a_2+\cdots$ of quaternions is convergent if $a_n\to 0$ and if the ratios $a_{n+1}a_n^{-1}$ lie in or on a four polyhedron which is contained in the closed unit four sphere, and which touches the sphere at one and only one point w, and $w\neq 1$.*

DERIVATION OF THE NORMAL FORM OF THE EQUATION OF THE STRAIGHT LINE

KENNETH MAY, Carleton College

The derivations of the normal form of the equation of the straight line which appear most frequently in elementary texts are based on the slope-intercept form, point-slope form, or on projection. The derivation from the intercept form, although less often found, has advantages of greater simplicity, symmetry and easy extension to three dimensions. It requires merely the observa-

tion that in all cases where the intercepts exist they are given by

$$a = p/\cos\alpha \quad \text{and} \quad b = p/\sin\alpha,$$

where p is the always positive length of the normal from the origin to the line and α is the positive angle through which the positive x-axis must be rotated to coincide with the positive direction on this normal. Substitution in the intercept form yields the normal form in a single step. Lines parallel to one of the axes or passing through the origin require, as usual, special treatment.

Similarly, the normal form of the equation of the plane in space may be derived from the intercept form by noting that each intercept (when it exists) is equal to the length of the normal from the origin to the plane divided by the corresponding direction cosine. The analogy may be emphasized by defining, for the two dimensional case, a second angle between the y-axis and the normal to the line.

These derivations are easy for the student to understand and remember. They facilitate unified treatment of plane and solid analytics in an elementary course without requiring explicit discussion of direction cosines in the plane. They lend themselves to efforts toward developing the student's appreciation of mathematical form in general and of the similarities between spaces of different dimensions in particular.

THE ELLIPSE AS A CIRCLE WITH A MOVING CENTER

F. H. YOUNG, Oregon State College

Our purpose is to examine the possibility of exhibiting the ellipse as the locus of a point on a rotating circle with a center moving on the focal axis. This condition may be satisfied if we can express the ellipse in the form $(x-x')^2+y^2=b^2$, for the symmetry of the figure requires that the radius be equal in length to b, half the minor axis. Throughout this discussion we shall assume that $a>b$.

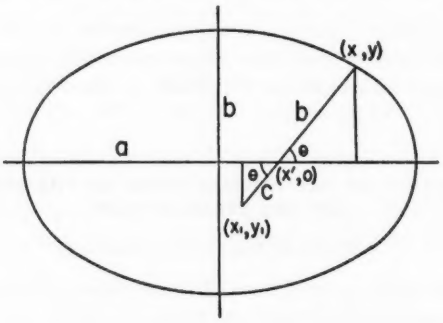

Fig. 1

In order to have x' move in such a way as to fit the ellipse $x^2/a^2+y^2/b^2=1$, we must let $x'=kx$ so that $x^2(1-k)^2/b^2+y^2/b^2=1$ where $b^2/(1-k)^2=a^2$, whence $k=(a-b)/a$.

Further, $x'=x-b\cos\theta$, but since $x'=[(a-b)/a]x$, then $x=a\cos\theta$. Also, $y=b\sin\theta$. Hence $x'=kx=(a-b)\cos\theta$. Therefore, $(x', 0)$, the center of the moving circle moves on the focal axis with simple harmonic motion over the range $-(a-b)\leqq x'\leqq(a-b)$ as θ increases constantly. Conversely, any such moving circle generates an unique ellipse, for the radius is of fixed length, b.

Let us now examine some of the properties revealed by this method of analysis of the ellipse. First, observe that the parametric equations $x=a\cos\theta$; $y=b\sin\theta$ can refer to the parameter θ, the angle the moving radius makes with the focal axis.

Consider next the situation arising when the radius is extended in the direction $\theta+\pi$ to give a point (x_1, y_1) lying the distance $b+c$ from (x, y) on the given ellipse. The point (x_1, y_1) then has the coordinates $y_1=-c\sin\theta$ and $x_1=x'-c\cos\theta=(a-b)\cos\theta-c\cos\theta=(a-b-c)\cos\theta$. Eliminating θ by squaring and adding, we obtain $y^2/c^2+x^2/(a-b-c)^2=1$, again an ellipse, provided $b+c\neq a$. In the particular case in which $b+c=a$, we find that $x=(a-b-c)\cos\theta=0$ and $y=(b-a)\sin\theta$, and as θ goes from 0 to 2π, we generate that section of the y-axis such that $-(a-b)\leqq y\leqq(a-b)$. If $b+c>a$, and we continue our assumption that $a>b$, then $c>a-b>0$; hence $|a-b-c|>c$ and our resulting locus is an ellipse whose major axis lies along the y-axis. It is interesting to note that if $c=a$, we have our original ellipse with the axes interchanged.

Notice now that the slope of the generating radius is ay/bx. Hence, the radius is normal to the ellipse only at the endpoints of the axes. This result gives the answer to the problem of the type of curve generated by a normal of fixed length N moving about the ellipse. For a fixed N, $b-a$ remains unchanged. Thus, x' moves in the same manner as for the original ellipse, and the radius is now $b+N$. If (x_1, y_1) lies on an ellipse, its distance from $(x', 0)$ is $b+N$, and the line

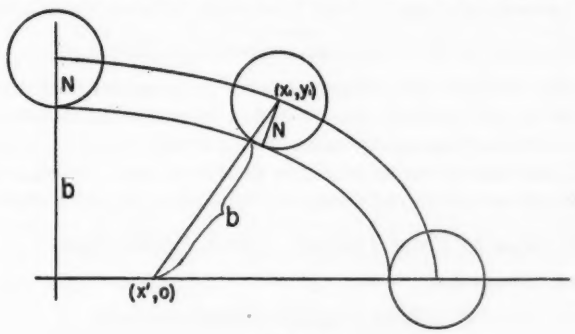

Fig. 2

joining these points is distinct from the normal. This requires the circle of radius N with center at (x_1, y_1) to be tangent to the ellipse externally in one place and to cut it in another, a contradiction. Hence, the curve generated by the moving normal is definitely not an ellipse. This can similarly be shown if N is measured in from the ellipse.

So far we have constrained b to be smaller than a. What results if we have our circle still moving on the x-axis, but $b>a$? This offers no difficulty, for from the relationship that $x'=(a-b)\cos\theta$, we can see that the center of the moving circle now must move from $[-(b-a), 0]$ to $[(b-a), 0]$, as θ increases positively, in describing the upper half of the ellipse whose equation is $x^2/a^2+y^2/b^2=1$, $b>a$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 806. *Proposed by Leo Moser, University of Manitoba*

Lewis Carroll once proposed the following problem.

"Two travellers spent from 2 o'clock till 9 in walking along a level road, up a hill, and home again; their pace on the level being x miles per hour, up hill y, and down hill $2y$. Find the distance walked."

In the original problem x and y were given integers. Deduce the solution to the original problem without *a priori* knowledge of what these integers are.

E 807. *Proposed by R. V. Andree, University of Wisconsin*

An elliptical endgate of a reservoir is to be mounted with the minor axis parallel to the water's surface and in such a manner that it will turn about a horizontal axis in the plane of the gate. Where should this axis be placed so that the gate will not tend to rotate in either direction when the water level is at a given distance above the top of the gate? Generalize to a gate of any shape.

E 808. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the number

$$N = 190004584615997768072777716631$$

is a perfect cube and that the twenty-eight numbers which are formed by cyclic permutations of its digits are all divisible by the cube root of N.

E 809. *Proposed by P. L. Chessin, New York, N. Y.*

Show that $\sum_{n=1}^{\infty} n^{-(n+1)/n}$ diverges.

E 810. *Proposed by J. H. Butchart, Arizona State College*

Find the sum of the infinite series consisting of the following terms: the radius of the unit circle, the arc intercepted by the central angle x, the arc of the involute corresponding to this arc, the arc of the involute constructed on the first involute, and so on indefinitely.

SOLUTIONS

The Case of the Playful Children

E 776 [1947, 339]. *Proposed by L. R. Ford, Illinois Institute of Technology*

"Are those your children that I hear playing in the garden?" asked the visitor.

"There are really four families of children," replied the host. "Mine is the largest, my brother's family is smaller, my sister's is smaller still, and my cousin's is the smallest of all. They are playing drop the handkerchief," he went on; "they prefer baseball but there are not enough children to make two teams. Curiously enough," he mused, "the product of the numbers in the four groups is my house number, which you saw when you came in."

"I am something of a mathematician," said the visitor, "let me see whether I can find the numbers of children in the various families." After figuring for a time he said, "I need more information. Does your cousin's family consist of a single child?" The host answered his question, whereupon the visitor said, "Knowing your house number and knowing the answer to my question, I can now deduce the exact number of children in each family."

How many children were there in each of the four families?

I. *Solution by Monte Dernham, San Francisco.* The scene changes to 221*B* Baker Street.

"Well, Holmes, how on earth did you deduce 5, 4, 3, 2 children in the respective families? How could you guess that the house number was 120?"

"My dear fellow," answered Holmes, lighting his clay pipe, "as you know, there were not enough children all told to make up two nines; so it was obvious from the first that the smallest of the four families could not include more than two children. After a little figuring, I soon discovered that for two children in the cousin's family our inquiry would be restricted to seven possible products, only one of which, 120, might also fit the case of a single child. Consequently," he went on to explain, "had the house number been anything but 120, the visitor, familiar with the number and confessedly 'something of a mathematician,' would have known *without the need of more information* whether the cousin had one or two children.

"It is equally clear," Sherlock Holmes continued, "that if the reply to his question had been 'Yes' our visitor would have been confronted with an 'ambiguous case'; you see, there might have been 8, 5, 3, 1 children in the respective families, or again, 6, 5, 4, 1. But, since he was enabled to 'deduce the exact number of children in each family,' it is plain that the answer he received was 'No,'

consistent with but a single admissible combination. Rather elementary, my dear Watson!"

II. *Solution by N. J. Fine, University of Pennsylvania.* Let N be the house number, $a<b<c<d$ the numbers of children in the four families. Since $a+b+c+d<18$, we have the bounds $1\leqq a\leqq 2$, $2\leqq b\leqq 4$. Since the visitor asked whether $a=1$ or 2, N must have at least two admissible representations, with $a=1$ and $a=2$.

Using $a=1$ we have $c+d\leqq 16-b$, whence

$$4cd = (c+d)^2 - (d-c)^2 \leqq (16-b)^2 - 1.$$

It follows that $cd\leqq 48, 42, 35$ for $b=2, 3, 4$ respectively, so that $N=bcd\leqq 140$.

Now using $a=2$, $N\geqq 2\cdot 3\cdot 4\cdot 5=120$. Hence $120\leqq N\leqq 140$. Now if N is not divisible by 5, $N\geqq 2\cdot 3\cdot 4\cdot 6=144>140$. If N is not divisible by 8, $N\geqq 2\cdot 3\cdot 5\cdot 6=180>140$. Thus N is divisible by 40, from which $N=120$.

There are two admissible factorizations of 120 with $a=1$:

$$120 = 1\cdot 4\cdot 5\cdot 6 = 1\cdot 3\cdot 5\cdot 8.$$

There is only one with $a=2$:

$$120 = 2\cdot 3\cdot 4\cdot 5.$$

If the answer to the visitor's equestion had been $a=1$, he could not have obtained a unique solution. Hence

$$a = 2, \quad b = 3, \quad c = 4, \quad d = 5.$$

Also solved by Murray Barbour, Max Beberman, Barney Bissinger, Ruth Campbell, J. L. Connors, R. E. Crane, J. S. Cromelin, J. H. Ferguson, C. W. Foard, C. O. Hines, William Kruskal, J. M. McLynn, Leo Moser, M. L. Oneil, Bart Park, C. F. Pinzka, W. M. Rust, Jr., G. W. Walker, Maud Willey, and R. H. Wilson, Jr.

The n-Kings Problem

E 777 [1947, 340]. *Proposed by C. R. Perisho, Nebraska Wesleyan University*

Find the number of permutations of n objects with the restriction that in no arrangement may an object be adjacent to either of its neighbors in the original order.

Note by Leo Moser, University of Manitoba. This problem is solved by I. Kaplansky in his paper *Symbolic solution of certain problems in permutations*, Bull. Amer. Math. Soc., vol. 50 (1944), pp. 911–913. The symbolic solution given there does not lend itself to expression by a simple explicit formula. As Kaplansky points out, the problem may be formulated as follows: In how many ways can n kings be placed on an n by n chess board, one on each row and column, so that no two attack each other?

Equal Shortest Paths

E 778 [1947, 340]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a given tetrahedron $ABCD$, find the point P in space such that the shortest paths separating P from each of the vertices A, B, C, D, after having touched the opposite faces BCD, CDA, DAB, ABC, are equal to each other.

Solution by G. W. Walker, Buffalo, N. Y. Locate A', the reflection of the vertex A across the plane of the opposite face, BCD, and locate B', C', and D' similarly. Then the shortest path from P to A touching the plane BCD will equal the distance PA'. (If O is the point of contact of the path with the plane, then $PO+OA=PO+OA'$ is a minimum when PA' is a straight line.) Thus the problem is simply to find the circumcenter P of the tetrahedron $A'B'C'D'$.

Editorial Note. It should be noted that the above solution tacitly assumes that the circumcenter of $A'B'C'D'$ is inside, or on a face, of $ABCD$. Under similar restrictions the corresponding problem in the plane is solved in an analogous fashion.

A Divisibility Problem

E 779 [1947, 340]. *Proposed by P. A. Pizá, San Juan, Puerto Rico*

Let x, y, z be three positive integers and set

$$a = x + y, \qquad b = x + z, \qquad c = x + y + z.$$

Show that for any prime exponent $p>2$,

$$(ab)^p - (cx)^p - (yz)^p$$

is divisible by the product $pabcxyz$.

Solution by N. J. Fine, University of Pennsylvania. Let $ab=A$, $cx=B$, $yz=C$. Then $A=B+C$. We have

$$\begin{aligned}
A^p - B^p - C^p &= (A-B)\sum_{r=0}^{p-1} A^{p-r-1}B^r - C^p \\
&= C\left\{\sum_{r=0}^{p-1} A^{p-r-1}B^r - C^{p-1}\right\} \\
&= C\left\{A^{p-1} + B^{p-1} + AB\sum_{r=1}^{p-2} A^{p-r-2}B^{r-1} - (A-B)^{p-1}\right\} \\
&= C\left\{A^{p-1} + B^{p-1} + AB\sum_{r=1}^{p-2} A^{p-r-2}B^{r-1}\right. \\
&\qquad \left. - \sum_{r=0}^{p-1}(-1)^r\binom{p-1}{r}A^{p-r-1}B^r\right\}
\end{aligned}$$

$$= ABC\left\{\sum_{r=1}^{p-2} A^{p-r-2}B^{r-1} - \sum_{r=1}^{p-2}(-1)^r\binom{p-1}{r}A^{p-r-2}B^{r-1}\right\}$$

$$= ABC\sum_{r=1}^{p-2} A^{p-r-2}B^{r-1}\left\{1 - (-1)^r\binom{p-1}{r}\right\}.$$

But, for $1 \leqq r \leqq p-1$ and p an odd prime,

$$\binom{p-1}{r} = \binom{p}{r} - \binom{p-1}{r-1} \equiv -\binom{p-1}{r-1} \pmod{p}.$$

Hence

$$\binom{p-1}{r} \equiv (-1)^r \pmod{p},$$

so that each factor

$$\left\{1 - (-1)^r\binom{p-1}{r}\right\}$$

is divisible by p, and the theorem follows.

Also solved by Murray Barbour, P. A. Clement, F. J. Duarte, Leo Moser, and G. W. Walker. Clement pointed out that a calculation, taking $x=y=z=1$, shows the necessity of requiring $p>2$.

Cutting Out a Lampshade

E 780 [1947, 340]. *Proposed by G. Pólya, Stanford University*

A lampshade has the shape of a frustum of a right circular cone. Its perimeter is P at the bottom, p at the top, and its slant height is s. Show that such a lampshade can be cut out in one piece from a rectangular sheet of paper with dimensions

$$P \quad \text{and} \quad s + p(P-p)/8s.$$

You can even save paper for a flap to glue the ends together, except in the limiting case where $P=p$, when not a bit of paper is wasted.

Solution by Joseph Rosenbaum, Milford School, Conn. The lampshade *cannot* always be cut out in one piece from the given rectangle. One would suspect this conclusion from a consideration of the extreme case where $p=0$ and $P=2\pi s$. Here the shade is a circular disc of radius s, and obviously cannot be cut out in one piece from the rectangle, one of whose dimensions is now equal to s.

In order to obtain the facts for non-extreme situations we shall consider the two cases $P-p \leqq \pi s$ and $P-p > \pi s$. It will be shown that *in the first case the construction of the problem is always possible, but in the second case the construction is possible if and only if*

$$(1) \qquad p/P \geqq \left[2\cos\left(\pi - \frac{P-p}{2s}\right)\right] \Big/ \left[\left(\frac{P-p}{2s}\right)^2 - 2\right].$$

Let us, for the present, consider the piece for the shade as a sector of a circular ring, with R and r as the outer and inner radii, and α as half the central angle. It is easily shown that for the first case, α is non-obtuse. When $0<\alpha<\pi/2$, the piece can be cut from a rectangle $ABCD$ as shown in Figure 1, the piece being $EFGHIJE$.

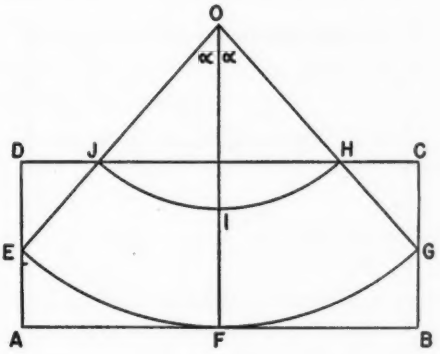

Fig. 1

Here we have

$$P = \text{arc } EFG > AB.$$

Also, from Maclaurin's expansion of $\cos x$,

$$f(\alpha) = \alpha^2/2 - 1 + \cos\alpha \geqq 0.$$

Now, setting $R=s+r$, $P=2\alpha R$, $p=2\alpha r$, we have

$$s + p(P-p)/8s - R + r\cos\alpha = rf(\alpha) \geqq 0,$$

or

$$s + p(P-p)/8s \geqq R - r\cos\alpha = BC.$$

Thus, for $0<\alpha<\pi/2$, the dimensions of the given rectangle are never less than those of $ABCD$, and the lampshade can be cut out in one piece in the shape of a sector of a circular ring. It is easily shown that the given rectangle similarly suffices for the extreme cases where $\alpha=0$ (that is, where $p=P$) and $\alpha=\pi/2$. Thus the first part of the italicized statement is established.

Let us now consider the second case and still take the piece as a sector of a circular ring. Here α is obtuse and the piece can be cut from a rectangle $ABCD$ as shown in Figure 2. The dimensions of the rectangle $ABCD$ are now $2R$ and $R+R\cos(\pi-\alpha)$. Hence, for the given rectangle to be adequate, it is necessary and sufficient that

$$s + p(P-p)/8s \geqq R + R\cos(\pi - \alpha).$$

Substituting, as above, for the several quantities, but leaving the second R intact, this becomes

$$r\alpha^2/2 \geqq r + R\cos(\pi - \alpha)$$

or

$$r/R \geqq [2\cos(\pi - \alpha)]/(\alpha^2 - 2).$$

This, expressed in terms of the given quantities, is the condition (1) above.

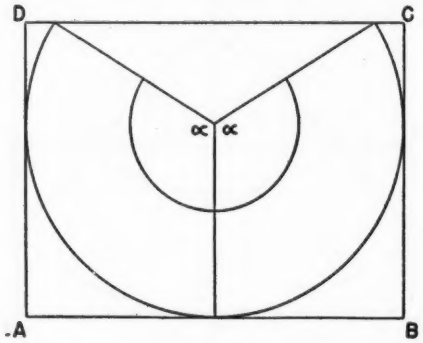

Fig. 2

It finally remains to show that the dimensions of the circumscribed rectangle $ABCD$ do not decrease when the piece which is cut out has any shape other than a sector of a circular ring. Toward this end we first observe that the general admissible shape for the piece in question is defined as follows:

Let M and N be any two points respectively on the lower and the upper circumferences of the frustum, and let L be any path, on the surface, joining M to N. If L has no points in common with the circumferences other than M and N, and if L does not cross itself, and if L is such that when the surface is cut along L and then flattened out, the resulting plane figure has no overlapping areas, then this plane figure is an admissible shape for the piece. (The last condition on L is necessitated by the fact that without it the piece cannot be cut from *any* plane sheet. That such pieces may exist is readily discovered by models.)

It is obvious that the above includes all admissible shapes, and it is further obvious that in any such shape a part of the perimeter of the piece will be the outer arc of the previous sector, and thus the dimensions of the corresponding rectangle cannot be less than those for the sector. This establishes the second part of the italicized statement.

Also solved, partially, by Ragnar Dybvik, R. B. Herrera, Irene Price, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4275 [1947, 601]. Corrected. *Proposed by Raymond Redheffer, Massachusetts Institute of Technology, Cambridge*

Let a_i represent any set of points in the complex plane, with sole limit-point at infinity, while b_i are any complex numbers. Prove there exists an integral function $f(z)$ such that $f(a_i)=b_i$.

4285. *Proposed by H. N. Davis, Victor Erikson, and Robert Nathans, United States Army*

A full reel of film (thickness τ) of original diameter A is being wound onto an empty spool of original radius a and rotating at a uniform angular velocity ω. How long does it take to unwind the first spool if its inner diameter is also a?

4286. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland*

Prove that

$$\int_0^\infty \frac{\cos x^2 - \cos x}{x}\,dx = \tfrac{1}{2}\gamma,$$

where γ is Euler's constant.

4287. *Proposed by C. R. Phelps, Rutgers University*

Show that for any given integer $k>1$, there are an infinite number of perfect kth powers which cannot be written as the sum of a prime and a kth power. (This disproves a conjecture of Hardy and Wright, *Introduction to the Theory of Numbers*, p. 19.)

4288. *Proposed by J. W. Campbell, University of Alberta*

In his Messenger Lecture on Probability (A. S. Eddington, *New Pathways in Science*, 1935, p. 121) Eddington referred to the following problem: If A, B, C, D each speak the truth once in three times (independently), and A affirms that B denies that C declares that D is a liar, what is the probability that D was telling the truth. He used the Exclusion Method of Solution and arrived at the numerical result 25/71.

Prove that the correct probability is 13/41, and that this is also the probability that each of A, B, and C told the truth.

4289. *Proposed by J. T. Culbertson, Southwestern University, Georgetown, Texas*

Let C be any finite class of n similar objects ($n>1$) and call any subclass of C a singular class if it contains only one of the n objects and call it a plural class if it contains more than one of the n objects. Also let any set of classes consisting of C and all the singular subclasses of C, together with any other subclasses of C (or none other at all) be defined as a classification of C. Thus any classification has r members where $n<r<2^n$, and there are $M=2^n-n-2$ subclasses of C, each of which may or may not belong to any particular classification (all the plural subclasses except C itself). Therefore there are 2^M classifications of C.

Let K be any classification of C and a and b any two objects in C. By interchanging a and b we get a classification K' which may or may not be identical with K. Let us call K' an interchange of K. Also let this relation be transitive, that is, if K, K' and K'' are any three classifications of C, and K' is an interchange of K, and K'' is an interchange of K', then K'' is an interchange of K. Then any two or more classifications are defined as structurally similar if and only if any one of them is an interchange of each of the others.

The set of all classifications structurally similar to any given classification is an α-set, e.g., when $n=3$ there are 4 α-sets.

The problem is: How many α-sets does C have; that is, how many structurally different sets of classifications are there for any such class of n objects?

SOLUTIONS

Squares Constructed Within A Triangle

3990 [1941, 214]. *Proposed by V. Thébault, Tennie, Sarthe, France*

Let A', B', C' be the centers of squares $BCA_1'A_2'$, $CAB_1'B_2'$, $ABC_1'C_2'$ constructed interiorly on the sides of triangle ABC with the centroid G and the angle V of Brocard. If cot $V=7/4$, show that: (1) The centers A'', B'', C'' of the squares constructed interiorly on the sides of $A'B'C'$ lie on a straight line through G. (2) The angle V' of Brocard of $A'B'C'$ is such that cot $V'=2$. (3) The straight lines joining A, B, C respectively to the midpoints of $A_1'A_2'$, $B_1'B_2'$, $C_1'C_2'$ are parallel. (4) The distance of the circumcenter from the orthocenter of the orthic triangle is equal to one-fourth of the perimeter of the last triangle.

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.** Letting S represent the area of triangle ABC we have

$$\overline{B'C'}^2 = \overline{AB'}^2 + \overline{AC'}^2 - 2\overline{AB'}\cdot\overline{AC'}\cos\overline{B'AC'} = (b^2 + c^2 - 2bc\sin A)/2$$
$$= (b^2 + c^2 - 4S)/2.$$

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* Translated and checked by O. J. Ramler, Catholic University of America, Washington, D. C.

If $(A'B'C')$ denotes the area of triangle $A'B'C'$,

$$(A'B'C') = (OB'C') + (OC'A') + (OA'B'),$$

where O is the circumcenter of ABC. Hence

$$\begin{aligned} 2(A'B'C') &= R^2 \sum (\cos C - \sin C)(\cos B - \sin B) \sin A \\ &= R^2 \sum (\sin 2A - \sin^2 A) = S(2 - \cot V), \end{aligned}$$

where R is the circumradius of ABC. The normal coördinates of A', B', C' are

$$\begin{aligned} &x_{A'} = a/2, \quad x_{B'} = b(\sin C - \cos C)/2, \quad x_{C'} = c(\sin B - \cos B)/2, \\ &y_{A'} = a(\sin C - \cos C)/2, \quad y_{B'} = b/2, \quad y_{C'} = c(\sin A - \cos A)/2, \\ &z_{A'} = a(\sin B - \cos B)/2, \quad z_{B'} = b(\sin A - \cos A)/2, \quad z_{C'} = c/2. \end{aligned}$$

Whence

$$x_{A'} + x_{B'} + x_{C'} = h_1, \qquad y_{A'} + y_{B'} + y_{C'} = h_2, \qquad z_{A'} + z_{B'} + z_{C'} = h_3,$$

where h_1, h_2, h_3 are the altitudes of ABC. Therefore, $A'B'C'$ has G for its centroid. Since $2(A'B'C') = S(2 - \cot V)$, it follows that $A'B'C'$ are collinear when $\cot V = 2$. This proves part (2), for the same relationships hold for triangle $A''B''C''$ in respect to triangle $A'B'C'$. Hence $\cot V' = 2$.

Now

$$\cot V' = 2 = \frac{\overline{A'B'}^2 + \overline{B'C'}^2 + \overline{C'A'}^2}{4(A'B'C')} = \frac{a^2 + b^2 + c^2 - 6S}{2S(2 - \cot V)}.$$

But $\cot V = (a^2+b^2+c^2)/4S$, hence $\cot V = 7/4$.

The midpoints of $A_1'A_2'$, $B_1'B_2'$, $C_1'C_2'$ have the normal coördinates

$$\begin{array}{lll} a, & a(\sin C - 2\cos C)/2, & a(\sin B - 2\cos B)/2, \\ b(\sin C - 2\cos C)/2, & b, & b(\sin A - 2\cos A)/2, \\ c(\sin B - 2\cos B)/2, & c(\sin A - 2\cos A)/2, & c. \end{array}$$

The lines joining the midpoints of $A_1'A_2'$, $B_1'B_2'$, $C_1'C_2'$ to A, B, C respectively meet in the point whose normal coördinates are given by

$$x(\sin A - 2\cos A) = y(\sin B - 2\cos B) = z(\sin C - 2\cos C).$$

These lines will be parallel when

$$\sum \sin A(\sin B - 2\cos B)(\sin C - 2\cos C) = 0$$

or

$$7 \sin A \sin B \sin C - 2 \sum \sin^2 A = 0,$$

which reduces to

$$a^2 + b^2 + c^2 = 7S, \qquad \cot V = 7/4.$$

If H_1, H_2, H_3 are the feet of the altitudes and H' is the orthocenter of $H_1H_2H_3$ (the orthic triangle of ABC) the normal coördinates of H' with respect to the triangle $H_1H_2H_3$ are given by

$$x \cos H_1 = y \cos H_2 = z \cos H_3.$$

The distance of H' to BC is proportional to

$$\frac{\cos H_2 + \cos H_3}{2 \cos H_2 \cos H_3 \sin (H_1/2)} = -\frac{\cos 2B + \cos 2C}{2 \cos 2B \cos 2C \cos A}$$
$$= -\frac{2 \cos (B + C) \cos (B - C)}{2 \cos 2B \cos 2C \cos A},$$

hence proportional to $\cos 2A \cos(B-C)$, since the equation of BC with respect to triangle $H_1H_2H_3$ is $y+z=0$. The actual normal coördinates of H' in triangle ABC are then readily found to be

$$x_{h'} = -R \cos 2A \cos (B - C), \qquad y_{h'} = -R \cos 2B \cos (C - A),$$
$$z_{h'} = -R \cos 2C \cos (A - B).$$

Now $\cos 2A \cos (B-C)$ can be represented as

$$-\tfrac{1}{2}[\sin A \textstyle\sum \sin 2A + \cos A(\sum \cos 2A - 1)], \quad \text{and}$$

$$\textstyle\sum \sin 2A = \dfrac{2S}{R^2}, \quad \text{and} \quad \sum \cos 2A - 1 = 2 - \dfrac{a^2 + b^2 + c^2}{2R^2}.$$

If x_k and x_0 are the distances of the Lemoine point K and of the circumcenter O of ABC from BC we have,

$$x_k = R \tan V \sin A, \qquad x_0 = R \cos A.$$

Hence

$$x_{h'} - x_0 = S \cot V(x_k - x_0)/R^2$$

and

$$x_{h'} - x_k = \left(\frac{S \cot V}{R^2} - 1\right)(x_k - x_0).$$

Therefore

$$\frac{H'O}{H'K} = \frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2 - 4R^2}, \qquad \frac{H'O}{KO} = \frac{a^2 + b^2 + c^2}{4R^2} = \frac{S \cot V}{R^2}.$$

Now by a known formula $OK = R \tan V\sqrt{\cot^2 V - 3}$ and

$$OH' = \frac{S\sqrt{\cot^2 V - 3}}{R}.$$

If cot $V = 7/4$, then

$$OH' = \frac{S}{4R}.$$

Now $2S/R$ is the perimeter of the orthic triangle. Hence OH' is one-eighth of this perimeter instead of one-fourth as stated in the proposal.

Second Lemoine Point

4223 [1946, 537]. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

In a tetrahedron the harmonic plane of the point L whose normal coördinates are proportional to the radii of the circumcircles of the triangles of the faces (second Lemoine point), coincides with the polar plane with respect to the circumsphere.

Solution by R. Goormaghtigh, Bruges, Belgium. Let $A_1A_2A_3A_4$ be the tetrahedron, a_{ij} the length of the edges A_iA_j, S_i and R_i the area and the circumradius of the face opposite to A_i.

The normal coördinates of L being proportional to $R_1, \cdots$, the barycentric coördinates are proportional to $S_1R_1, \cdots$, or $a_{23}a_{34}a_{42}, \cdots$.

The barycentric equation of the circumsphere being

$$\sum\nolimits^{2} a_{ij}x_ix_j = 0,$$

the polar plane of L is

$$a_{12}a_{13}a_{14}x_1 + \cdots = 0$$

or

$$x_1/a_{23}a_{34}a_{42} + \cdots = 0$$

and this is the equation of the harmonic plane of L.

Also solved by M. R. Blanchard.

Central Ellipse of Inertia

4225 [1946, 594]. *Proposed by H. F. Sandham, Trinity College, Ireland*

There are given N points in a plane, the roots of $f(z)=0$. A conic is constructed with center at the mean center, and is such that the sum of the squares of the perpendiculars from the points on to a line through the center is $N(N-1)$ times the square of the perpendicular from the center on to a parallel tangent. Prove that the foci are given by $f^{(N-2)}(z)=0$.

Solution by Fritz John, New York University. Let the N points

$$z_k = x_k + iy_k \qquad k = 1, \cdots, N$$

be the roots of the equation

$$f(z) = z^N - Az^{N-1} + Bz^{N-2} - \cdots = 0.$$

Without restriction of generality we can assume that the mean center O of the points is at the origin of the coördinate system. Then

$$\sum_k z_k = A = 0,$$
$$\sum_k z_k^2 = A^2 - 2B = -2B. \tag{1}$$

Let E be the ellipse with center O such that the sum of the squares of the perpendiculars from the z_k to a line through O is $N(N-1)$ times the square of the distance of the line from a parallel tangent of E. Let $ux+vy=1$ be the equation of a tangent of E. Then the distance of that tangent from the parallel diameter is $(u^2+v^2)^{-1/2}$, and the distance of a point (x, y) from that diameter is

$$\left| \frac{ux + vy}{\sqrt{u^2 + v^2}} \right|.$$

The definition of E can then be written

$$\sum_k (ux_k + vy_k)^2 = N(N-1), \tag{2}$$

which can be interpreted as the equation of E in tangential coördinates.

The foci of E are the intersections of conjugate imaginary isotropic tangents of E. Here the isotropic tangents are those of slope $\pm i$. For an isotropic tangent of E of slope $+i$ we have $v=iu$, and hence from (2)

$$u^2 \sum_k z_k^2 = N(N-1). \tag{3}$$

The equation of that isotropic tangent is then $u(x+iy)=1$. Its intersection with its conjugate imaginary tangent, $\bar{u}(x-iy)=1$, is then represented by the complex number

$$z = x + iy = 1/u. \tag{4}$$

The two foci of E are then seen to be represented by the reciprocals of the roots of the quadratic equation (3) and are therefore the roots of

$$z^2 = \frac{1}{N(N-1)} \sum_k z_k^2,$$

which, by use of (1), reduces to

$$z^2 + \frac{2B}{N(N-1)} = 0,$$

or

$$f^{(N-2)}(z) = \frac{N!}{2} z^2 + (N-2)!B = 0,$$

so that the proof is complete.

From its definition E can be seen to be essentially the central ellipse of inertia of the z_i. (More precisely, the two ellipses are homothetic from O.) The statement of the problem can then be reduced to a theorem of Lucas, quoted in Appell, *Mecanique Rationelle*, t. 2, p. 16, ex. 13, and which is reproduced here freely in the notation used above: *The roots of* $f^{(N-2)}(z)=0$ *lie on the major axis of the central ellipse of inertia of unit masses distributed over the roots of* $f(z)=0$. *The difference of the squares of the radii of gyration of the masses about the principal axes of inertia is* $(N-1)/4$ *times the square of the distance of the two roots of* $f^{(N-2)}(z)=0$.

Also solved by H. E. Fettis and J. H. Simester.

Regions in which Polynomials Have Absolute Value Not Exceeding Unity

4229 [1947, 49]. *Proposed by Paul Erdös, Syracuse University*

Let $f(z)=z^n+\cdots+a_n$, $g(z)=z^m+\cdots+b_m$ be two polynomials. Denote by A the region where $|f(z)|\leqq 1$ and by B the region where $|g(z)|\leqq 1$. Prove that A cannot properly contain B.

Solution by Tibor Radó, Ohio State University. If $P(z)=z^k+\cdots$ is a polynomial of degree $k\geqq 1$, we shall denote by $D(P)$ the open set on which $|P(z)|>1$, and by $D^*(P)$ the (obviously unique) unbounded component of $D(P)$. Note that $D^*(P)$ contains a whole neighborhood of $z=\infty$, and that the frontier of $D^*(P)$ is not empty. Our problem is then equivalent to that of proving that $D(f)$ cannot be a proper subset of $D(g)$.

Let us now assume merely that $D(f)$ is a subset of $D(g)$ (not necessarily proper), in symbols $D(f)\subset D(g)$. Then clearly $D^*(f)\subset D^*(g)$. Consider the auxiliary function $\Psi(z)=F(z)/G(z)$ for $z\in D^*(f)$, where $F(z)=[f(z)]^m$ and $G(z)=[g(z)]^n$. Clearly $D(F)=D(f)$ and $D(G)=D(g)$. Then $\Psi(z)$ is analytic in $D^*(f)$, and, since $|G(z)|\geqq 1$ on the frontier of $D^*(f)$, $|\Psi(z)|\leqq 1$ on the frontier of $D^*(f)$. On the other hand, $\Psi(z)\to 1$ for $z\to\infty$. By the maximum-modulus theorem it follows that $\Psi(z)\equiv 1$ and hence $F(z)\equiv G(z)$ in $D^*(f)$ and consequently $F(z)\equiv G(z)$ throughout the whole plane. It follows that $D(f)=D(g)$ as required.

Note by Marshall Hall, Ohio State University. If $(m, n)=d$ and $m=dm_1$, $n=dn_1$, the above proof shows that $[f(z)]^{m_1}\equiv[g(z)]^{n_1}$; and since m_1 and n_1 are relatively prime, it follows from the uniqueness of factorization of polynomials over a field that there is a polynomial $h(z)$ such that $f(z)=[h(z)]^{n_1}$, $g(z)=[h(z)]^{m_1}$. Thus the relation $D(f)\subset D(g)$ holds if and only if $f(z)$ and $g(z)$ are positive integral powers of the same polynomial $h(z)$.

Also solved by Robert Breusch, Michael Golomb, Fritz Herzog, and the proposer.

Integers Reproduced in the Right-hand Digits of Their Squares

4230 [1947, 49]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In every system of numeration in which the base B is divisible by two or more

distinct primes but not by 3 or any prime of the form $6k+1$, the numbers which have the property that they are reproduced in the right-hand digits of their squares are the same as those which are reproduced at the right of their fourth powers. (For example: $B=10$, $76^2=5776$, $76^4=33362176$.)

Solution by Fritz Herzog, Michigan State College. The proposal is a special case (corresponding to $p=3$) of the following more general theorem.

Let p be an odd prime and let B be an integer greater than unity, not divisible by p nor by any prime of the form $pk+1$. In the system of numeration with B as base, a number is reproduced in the right-hand digits of its $(p+1)$st power if and only if it is reproduced in the right-hand digits of its square.

Let x be a number of m digits. We then have to show that $x^{p+1}\equiv x \pmod{B^m}$ if and only if $x^2\equiv x \pmod{B^m}$. It obviously suffices to show that the first congruence implies the second. Let q be any prime factor of B and let q^s be the highest power of q that divides B. From the first of the above congruences we have

$$x^{p+1} \equiv x \pmod{q^{sm}}, \tag{1}$$

and it remains to show that, for all prime factors q of B,

$$x^2 \equiv x \pmod{q^{sm}}. \tag{2}$$

Now if $(x, q)=1$, then we obtain from (1) the relation $x^p\equiv 1 \pmod{q^{sm}}$; hence the exponent of x modulo q^{sm} is either p or 1. But by hypothesis, q is not congruent to 0 or 1 (mod p), hence p cannot divide $\phi(q^{sm})=q^{sm-1}(q-1)$ and,therefore, we must have $x\equiv 1 \pmod{q^{sm}}$, which implies (2). On the other hand, if $x\equiv 0 \pmod{q}$ we write (1) in the form $(x^p-1)x\equiv 0 \pmod{q^{sm}}$ and, since in this case $(x^p-1, q^{sm})=1$, we have $x\equiv 0 \pmod{q^{sm}}$, which again implies (2).

Also solved by Free Jamison and the Proposer.

Editorial Note. The Proposer remarks that the condition that B be composite is not necessary for the proof, but if B is prime there are no numbers less than B^m except 0 and 1 which have the desired property. The proposed problem was suggested by a paper appearing in *Mathesis*.

Curves Having a Property of the Parabola

4231 [1947, 49]. *Proposed by Paul Nemenyi, Washington State College*

Show that the parabola $y=ax^n (a\neq 0,\ n>0)$ has the following property: If through the vertex any ray is drawn, the ratio of the area of the segment to that of the largest inscribed triangle is independent of the direction of the ray. Are there other curves with the same property?

Solution by Arthur Breusch, Amherst College. I. If $y=f(x)=ax^n+bx$, $n>0$, $n\neq 1$, then the area between the curve and the chord joining $(0, 0)$ to (x, y) is

$$\pm\left(\frac{1}{2}xy-\int_0^x y\,dx\right)=\pm\frac{n-1}{2(n+1)}ax^{n+1}.$$

The area of the largest inscribed triangle is

$$\pm \tfrac{1}{2}[x_1 f(x) - x f(x_1)],$$

where $x f'(x_1) = f(x)$. Therefore $x_1/x = n^{1/(1-n)}$, and the area of the triangle is

$$\pm \frac{(n-1)ax^{n+1}}{2n^{n/(n-1)}}.$$

The signs will be both + or both − according as n is greater than or less than 1. The desired ratio is therefore

$$\frac{n^{n/(n-1)}}{n+1}.$$

II. Any curve given by

$$\alpha x + \beta y = (\gamma x + \delta y)^n, \qquad n > 0 \tag{1}$$

will also have the property; for a rotation of coördinate axes will reduce this equation to the form $\bar{y} = a\bar{x}^n + b\bar{x}$.

It can be shown that, under certain restrictions, if $y = f(x)$ is a function whose graph has the desired property, then y is given by a relation of the type (1).

III. Lemma 1: *If the curve given by $y = f(x)$ has the desired property, and if $f'(x)$ and $f''(x)$ exist in $0 < x \leqq h$, then $f(x)$ must satisfy in $0 < x \leqq h$ the functional equation*

$$f'\left[\frac{\int_0^x x^2 f''(x)\,dx}{k\int_0^x x f''(x)\,dz}\right] = \frac{f(x)}{x}, \tag{2}$$

where k is a constant.

Proof: Given

$$x f(x) - 2\int_0^x f(x)\,dx = k[x_1 f(x) - x f(x_1)], \tag{A}$$

with

$$x f'(x_1) = f(x), \text{ and } k \text{ constant}. \tag{B}$$

Differentiating (A) with respect to x, we have

$$x f'(x) - f(x) = k[x_1 f'(x) - f(x_1)] + k[f(x) - x f'(x_1)]\,dx_1/dx,$$

which by means of (B) reduces to

$$x f'(x) - f(x) = k[x_1 f'(x) - f(x_1)].$$

From this last equation and equation (A) we can compute

$$x_1 = \frac{x^2f'(x) - 2xf(x) + 2\int_0^x f(x)dx}{k[xf'(x) - f(x)]} = \frac{\int_0^x x^2f''(x)dx}{k\int_0^x xf''(x)dx} \equiv \frac{\phi(x, f)}{k},$$

where we have introduced the symbol $\phi(x, f)$ to simplify the notation. Placing this value of x_1 in equation (B) we obtain (2).

IV. Lemma 2: *If $f(x)$ satisfies* (2), *then $g(x) = af(x) + bx$ does likewise.*

Proof: $g''(x) = af''(x)$, therefore $\phi(x, g) = \phi(x, f)$. Also $g' = b + af'$. Therefore

$$g'[\phi(x, g)] = b + af'[\phi(x, f)] = b + a\frac{f(x)}{x} = \frac{g(x)}{x}.$$

Thus if $y = f(x) = Ax + Bx^r + Cx^s + \cdots$ satisfies (2), then $\bar{y} = \bar{f}(x) = x^r + bx^s + \cdots$ does likewise, with $b = C/B$, $r \neq 1$, $s \neq 1$.

V. Lemma 3: *If $\bar{f}(x)$ satisfies* (2) *and is a sum of more than one non-linear terms: $\bar{f}(x) = x^r + bx^s +$ terms of higher degree with $r < s$, $r \neq 1$, $s \neq 1$, $b \neq 0$, then s must equal $2r - 1$.*

Proof: By direct computation we obtain

$$\phi(x, \bar{f}) = \frac{\frac{r}{r+1}x + b\frac{s(s-1)}{(s+1)(r-1)}x^{s-r+1} + \cdots}{1 + b\frac{s-1}{r-1}x^{s-r} + \cdots}$$

$$= \frac{r}{r+1}x + b\frac{(s-1)(s-r)}{(r+1)(s+1)(r-1)}x^{s-r+1} + \cdots.$$

Substituting this in (2) and simplifying, we have

$$\frac{r}{k^{r-1}}\left[\left(\frac{r}{r+1}\right)^{r-1}x^{r-1} + (r-1)\left(\frac{r}{r+1}\right)^{r-2}b\frac{(s-1)(s-r)}{(r+1)(s+1)(r-1)}x^{s-1} + \cdots\right]$$

$$+ \frac{bs}{k^{s-1}}\left(\frac{r}{r+1}\right)^{s-1}x^{s-1} + \cdots = x^{r-1} + bx^{s-1} + \cdots.$$

Comparing coefficients, we find

$$\frac{r^r}{k^{r-1}(r+1)^{r-1}} = 1, \qquad \frac{r^{r-1}}{k^{r-1}(r+1)^{r-1}}\frac{(s-1)(s-r)}{s+1} + \frac{sr^{s-1}}{k^{s-1}(r+1)^{s-1}} = 1.$$

Upon eliminating k and simplifying we obtain

$$(2r + 1 - s)r^{(s-r)/(r-1)} - (s+1) = 0.$$

For a fixed positive value of $r \neq 1$, call the left member of this equation $g(s)$.

Then

$$g'(s) = r^{(s-r)/(r-1)}\left[-1+\frac{2r+1-s}{r-1}\log r\right]-1,$$

$$g''(s) = r^{(s-r)/(r-1)}\left[-2+\frac{2r+1-s}{r-1}\log r\right]\frac{\log r}{r-1}.$$

Since the quantities in brackets are linear in s, therefore $g''(s)$ can vanish only once, $g'(s)$ only twice, and $g(s)$ only three times for $-\infty<s<+\infty$; but it is easily seen that $g(r)=g(1)=g(2r-1)=0$, and since $r<s$ and $s\neq 1$, s must be $2r-1$.

Corollary: *If* $\bar{f}(x)=x^r+bx^s+$*terms of higher degree, satisfies* (2), *and if* $r<s$, $s\neq 1$, *and* $r<1$, *then* b *must be* 0; for $2r-1$ would be less than r.

VI. Theorem: *If* $\bar{f}(x)=x^r+\cdots$ $(r\neq 1)$ *has no linear term and satisfies* (2), *and if* $\bar{f}(x)$ *is either regular in the neighborhood of* $x=0$, *or of the form,* x^r *times a regular function, with* $r>0$, *then* $y=a\bar{f}(x)+bx$ *is connected with* x *by a relation of the form* (1).

Proof: If $r<1$, then $\bar{f}(x)$ has only the term x^r, and therefore $y=ax^r+bx$, which is of the form (1). If $r>1$, then $\bar{y}=x^r(1+\cdots)=x^r$ times a regular function of x. Then $\bar{y}^{1/r}=x(1+\cdots)=$a regular function of x with non-vanishing derivative at $x=0$. Therefore x is also a regular function of $\bar{y}^{1/r}$:

$$x = g(\bar{y}) = \bar{y}^{1/r} + \text{higher powers of } \bar{y}.$$

Since $g(\bar{y})$ must satisfy equation (2), and since $1/r<1$, the only other possible term is a linear one:

$$x = \bar{y}^{1/r} + \beta\bar{y}, \qquad (x-\beta\bar{y})^r = \bar{y},$$

and finally

$$y = a\bar{y} + bx = a\left(x-\beta\frac{y-bx}{a}\right)^r + bx,$$

which is again of form (1).

VII. If $f'(x)$ does not exist everywhere in $0<x\leqq h$, then its graph may have the prescribed property even though x and y are not connected as in (1). For example, $y=|x-1|-1$.

Also solved (first part) by R. P. Peterson, Jr., and the Proposer.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Plane Trigonometry. By E. B. Mode. New York, Prentice-Hall, Inc., 1947. 10+216 pages. $2.40.

This enlargement and revision of an earlier lithoprinted work contains several features that should make it appeal to students and teachers alike. One of these is the set of three supplementary chapters on Approximate Computation, Logarithms, and The Slide Rule. The author refers frequently to the first two of these chapters and suggests that they be taken up before the main body of the text, and it might have been preferable to place them at the beginning of the book.

In addition to an adequate treatment of the solution of triangles, the author has indicated various other applications of the trigonometric functions. His introductory chapter contains a discussion of angles measured in degrees, radians, and mils, and radian measure is encountered frequently in the following chapters. The trigonometric functions are defined for the general angle and followed by the definitions for the angles of a right triangle. An unusual feature of the book is the use of the concept of even function and odd function in the derivation of several of the properties of the trigonometric functions.

The good features of this book are somewhat counterbalanced by a few undesirable ones. At least twice one finds "sin" at the end of one line and "θ" (or "β") at the beginning of the next. The section entitled "A common error" might better have been omitted. It seems unfortunate to suggest such a possibility and the explanation given is far from convincing.

E. S. SOKOLNIKOFF

Proceedings of the First Canadian Mathematical Congress. The University of Toronto Press, 1946. 44+367 pages. $3.25.

The First Canadian Mathematical Congress was held in Toronto during the week beginning on June 17, 1945. The papers presented at the meetings of this Congress, the program, the list of those who registered, and the Minutes of the General Meeting are published in the *Proceedings*.

The Congress encompassed a wide range of subjects. Four papers were devoted to a *Discussion of Secondary School Mathematics*, five to a *Symposium on Statistics*, four to a *Discussion of Engineering Mathematics*, two to a *Discussion on Research and Graduate Work in Canada* and six to *Short Research Papers*. Of the other papers presented, nineteen in number, thirteen were chiefly expository in nature and dealt with topics in both pure and applied mathematics. The remaining six were less technical in scope and might be characterized as dealing with the role of mathematics in various educational programs.

In view of the wide range of topics covered, and the interesting way in which they are presented, these Proceedings should be of interest to a wide circle of readers.

H. P. EVANS

Intermediate Algebra for Colleges. By E. B. Miller. New York, the Ronald Press Co., 1947. 10+361 pages. $2.50.

In every respect, this is a traditional, elementary algebra text. The recent trend to bring some of the basic concepts from foundations and abstract algebra into the educational picture at an ever earlier stage has been disregarded completely in this text. In the form, as well as in the content of the book, there appears to be no very marked liaison with advanced courses.

The author's aims include a careful and accurate exposition, with no sparing of detail. These aims seem for the most part to have been achieved. Detail, especially, has in fact become a nemesis. For example, the policy of including a multitude of negative hints may serve only to befuddle the student and to increase his already great capacity for inventing wrong procedures. Furthermore, the emphasis on techniques (which is heralded in the preface) has taken the form of cookbook-like sets of rules. This must sacrifice concept for the sake of method, and conceal the fact that algebra is based on a compact set of principles. Is the student then likely to reason his way through a problem? I fear that, on the contrary, he will have been encouraged to seek a pigeonhole in which the author has obligingly stored instructions for the situation at hand.

Certain features of the book might be attractive to some pedagogical tastes. For example, in the interests of plausibility, each formal proof is preceded by the treatment of a special case, and certain topics dealing with negative numbers are accompanied by counterparts from everyday experience, thus serving to shield the student from their frightening abstract character. "Story" problems are given considerable play throughout the book. In one chapter, there is a classified treatment of digit problems, "How old is Mary?" problems, coin problems, lever problems, work problems, and several other standard types.

The chapter on equations and their solution is quite well done.

This book seems to be directed at the student who must be nursed along, who finds logical thinking difficult, and who has little resourcefulness or imagination.

G. F. ROSE

Éléments de Calcul Infinitésimal. By Adrien Grosrey. Paris, Gauthier-Villars, 1945. 192 pages. 280 Fr.

This book is intended for use by young technical students in the cantonal schools of Switzerland rather than for college or university students. It is, nevertheless, on about the same level as the majority of calculus texts used in colleges in the United States. The treatment is much more concise, however, and the exercises are fewer in number and appear only at the ends of the chapters.

Fundamental concepts are treated quite briefly and the main emphasis is on techniques and applications rather than theory. Limits and continuity, for example, are disposed of in the first chapter of ten pages. The definition of a limit given in this chapter is unsatisfactory since, in common with many calculus texts, the "limit of a variable" is defined without reference to either a sequence or a function. The usual theorems on limits are stated without proof.

The second chapter disposes of the derivative, rules for differentiation, and differentiation of the elementary functions—all in fourteen pages. The succeeding chapters cover all the topics usually taken up in courses in elementary calculus plus brief discussions of Fourier series, curvilinear integrals, and the graphical solution of equations.

Many of the demonstrations given in this text should be regarded only as making plausible the theorems which they purport to prove. Perhaps one should not be overly critical of this, however, in view of the purpose of the book and since the author admits freely in the preface that his exposition contains gaps and that some of the demonstrations are lacking in rigor.

H. P. EVANS

NEW BOOKS RECEIVED

College Algebra. By F. S. Nowlan. New York, McGraw-Hill Book Co., 1947, 14+371 pages. $3.00.

Intermediate Algebra. By R. S. Underwood, T. R. Nelson, and S. Selby. New York, The Macmillan Company, 1947. 7+283 pages. $2.60.

Five-figure Tables of Trigonometrical Functions. Prepared by H. M. Nautical Almanac Office. London, 1947. $4.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

Mathematics Club, Adelphi College

The projects for the last years had two aims. One was to give the mathematics majors insights into the various professions which are open to them. Representatives of the different professions spoke to the club in successive meetings and answered the questions which arose. The second aim was to arrange joint meetings with the mathematics clubs of neighbor colleges and universities. Since Adelphi College lies within the metropolitan area of New York City there are many contacts within easy reach.

The outstanding events of the last academic year were two joint meetings

with the *David Eugene Smith Mathematics Club* of Teachers College of Columbia University. In December the Adelphi *Mathematics Club* went to Teachers College and Professor H. von Baravalle gave his demonstration of moving skew curves with string models. In the spring the Teachers College *Mathematics Club* came to Adelphi and Professor W. D. Reeve spoke on the topic, *The future of mathematics education in America.*

The officers for 1945–46 were: President, Miss Alice Lawson; Secretary-Treasurer, Miss Agnes Cook. The newly elected officers for 1946–47 are: President, Miss Elsie Novy; Secretary-Treasurer, John Brull; Faculty Advisor, Professor H. V. Baravalle.

Pi Mu Epsilon, Ohio State University

During the Winter and Spring Quarters, *Ohio Alpha* chapter sponsored a series of five informal meetings, at which topics intended to be of interest especially to undergraduate mathematics students were discussed. Topics and speakers were:

Infinite sequences, by Professor Tibor Rado
A problem on squares, by W. R. Scott
The isoperimetric problem, by Professor E. J. Mickle
Certain geometric constructions, by Dr. L. H. Miller
Infinity, by Dr. R. L. Swain.

Average attendance at these meetings was approximately 50, most of them freshman and sophomores.

At a special meeting in February, problems comprising the Rasor Scholarship Examination, open to students not past first quarter calculus, were discussed by Professor Marshall Hall, Jr., and by Professor Mickle. Winners were announced, and B. L. Stradley, Vice President of Ohio State University, presented the awards, including a first prize of $50, won by Donald R. Kibbey.

In May the chapter sponsored a competition open to students not higher than third quarter freshman level. A committee consisting of Professor Hall, Mr. Herbert C. Parrish, and Mr. H. M. Fliess devised an examination. At the last meeting of the year, the problems constituting the examination were discussed by Mr. Parrish and Mr. Byron B. Dressler. Professor Hall announced the winners and presented the awards, including a first prize of $35 and a second prize of $15 won by Richard Rubenstein and Donald Ray McElwain, respectively.

At the annual initiation in May, Professor E. L. Pitcher of Lehigh University spoke, his topic being Quadratic Analysis. A total of 32 new members were initiated. At the banquet following the initiation, Professor Pitcher spoke briefly on *The ham sandwich problem.*

Officers for 1946–47 were: Director, William R. Scott; Vice-Director, Landon A. Colquitt; Secretary, D. Ransom Whitney.

Officers elected for 1947–48 are: Director, Ray E. Kidder; Vice-Director, Byron B. Dressler; Secretary, Faye Mozelle Rankin.

Mathematics Club, Case Institute of Technology

Five meetings were held during the year 1946–47; the speakers and their topics are as follows:

The harmonic analyser, by Professor S. W. McCuskey
Prime numbers, by B. Herzog
Numerical methods, by Professor O. E. Brown
Summability, by Professor L. J. Green
From Zeno's paradox to pursuit curves in modern warfare, by Dean Emeritus T. M. Focke.

The President of the Club was B. Herzog and the Faculty Adviser was Professor Max Morris.

Mathematics Club, Oberlin College

Bi-monthly meetings were held in the Physics building, the new location of the Mathematics Department. As had been the custom in former years, tea and cookies were served at the beginning of each regular meeting. Programs consisted of the following speeches:

The Möbius strip, by Ruth Berger
Linkages and straight lines, by Robert Graves
Hyperspace, by Frank Marzocco
Non-Euclidean Geometry, by Peter Manos
Soap bubbles, by Nancy Lowell
Opportunities for careers in mathematics, by Artha Jean Burington
Nomograms, by Margaret Waugh
Postulates of relativity, by Lester Arnold
Diphantine analysis, by Jean Wyre
Continued fractions, by Charlotte Peters
Postulational systems, by Elizabeth MacKay
Derangements, by Rosalind Monastersky.

In addition to the twelve regular meetings, two special meetings were held. Professor and Mrs. R. W. Wagner invited the Club to their home for a Christmas party. Mathematical charades drew dramatic and mathematical ingenuity into peculiar combinations. The annual banquet was open to all interested in mathematics. Professor L. M. Graves of the University of Chicago discussed *The spectra of proof*. Eighty guests attended the banquet.

Officers for 1946–47 were: President, Charlotte Peters; Vice-President, Frank Marzocco; Secretary-treasurer, Ruth Berger; Social Chairman, Artha Jean Burington; Publicity Chairman, Rosalind Monastersky; Faculty Advisor, Professor R. W. Wagner.

Officers elected for 1947–48 are: President, Lester Arnold; Vice-President Ruth Berger; Secretary-treasurer, Mary Wright; Social Chairman, Rosalind Monastersky; Publicity Chairman, Quentin Darmstadt; Faculty Advisor, Professor E. P. Vance.

Mathematics Club, Stanford University

The first postwar meetings of the Stanford *Mathematics Club* was held in February, 1947. It was decided that the Club should meet every three or four weeks, and that, in general, at each meeting an elementary and a more advanced talk should be given. Three meetings were held, at which the following papers were presented:

A theorem of Mertens, by Professor H. M. Bacon

Some problems in aerodynamics, by Professor J. G. Herriot

Application of geometrical optics to the calculus of variations, by Dr. R. Weinstock

Infinite sets, by Burnett Meyer

Geometric maxima and minima, by George Crane.

On May 17, a picnic was held at Searsville Lake with an attendance of about forty. During the year a new constitution was drawn up and adopted.

The officers for the year were: President, Albert Novikoff; Secretary-treasurer, Burnett Meyer; Faculty Adviser, Professor J. G. Herriot.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The Annual Meeting of the National Council of Teachers of Mathematics will be held in Indianapolis, Indiana, on April 2–3, 1948. Headquarters will be at the Hotel Claypool.

THE MATHEMATICS MAGAZINE

The Mathematics Magazine desires to call attention to a series of articles which will begin in its March–April issue. These articles will describe the general content of various courses in mathematics in terms that can be easily followed by the general reader. Articles on the more advanced courses will require of the untrained reader the reading of the preceding articles of the series.

The subscription price of The Mathematics Magazine is $3.00. Subscriptions should be sent to Professor Glenn James, University of California, Los Angeles 24, California.

TENTH INTERNATIONAL CONGRESS OF PHILOSOPHY

The Tenth International Congress of Philosophy will be held at Amsterdam, August 11–18, 1948. Information about the Congress and application forms for

membership may be obtained from Professor Dr. E. W. Beth, Bern Zweerskade 231, Amsterdam Z, Holland.

STATISTICAL SESSIONS AT ALABAMA POLYTECHNIC INSTITUTE

Professor G. W. Snedecor, President of the American Statistical Association and Research Professor of Statistics at Iowa State College, will be Visiting Research Professor of Statistics at Alabama Polytechnic Institute during the Spring Quarter, from March 22 to June 4, 1948. Professor Snedecor will lecture on Statistical Experimental Design and will be available for statistical consultations.

The newly formed Statistical Laboratory at A. P. I. will also offer a course in Survey Sampling during the Spring Quarter to be taught by the Director, Professor T. A. Bancroft. Conferences in applied statistics for research workers in the lower southeastern states are being scheduled during the time of Professor Snedecor's visit.

PERSONAL ITEMS

Dr. Leo A. Aroian of Hunter College has been promoted to an assistant professorship.

Associate Professor H. G. Ayre of Western Illinois State College has been promoted to a professorship and has become Director of the General College Division.

Assistant Professor S. Louise Beasley of Drury College has been appointed to an assistant professorship at Carleton College.

Assistant Professor W. R. Callahan of Northeastern University has been appointed to an assistant professorship of applied mechanics at New York University.

Dr. W. B. Caton of the University of Maine has been appointed to an assistant professorship at Washington State College.

Assistant Professor Francis L. Celauro of Lehigh University has been appointed Assistant Professor of Mathematics at the Newark College of Engineering.

Professor W. W. S. Claytor of Hampton Institute has been appointed to an associate professorship at Howard University.

Dr. J. D. Elder of the University of Michigan has been appointed to an associate professorship at St. Louis University.

Professor William Findlay of McMaster University has retired with the title of Professor Emeritus.

Dr. G. E. Forsythe of Boeing Aircraft Company has been appointed to an assistant professorship in meteorology at the University of California.

Dr. W. C. G. Fraser of Dartmouth College has been appointed to an assistant professorship at Rensselaer Polytechnic Institute.

Dr. H. E. Goheen of the Office of Research and Inventions, Navy Department, has been appointed to an assistant professorship at the University of Delaware.

Assistant Professor W. W. Gutzman of the United States Naval Postgraduate School has been appointed to a professorship at the University of South Dakota.

Assistant Professor Coleman Herpel of Pennsylvania State College, Altoona, Pennsylvania has been promoted to an associate professorship.

Assistant Professor I. M. Hostetter of Oregon State College has been promoted to an associate professorship.

Professor A. E. Johns of McMaster University has been appointed Head of the Department of Mathematics.

Mr. Sidney Kaplan of the Bureau of the Census has joined the Numerical Analysis Section of the Naval Ordnance Laboratory. In addition, he is serving as part-time lecturer in mathematics at the Catholic University of America.

Associate Professor P. E. Lewis of Oklahoma Agricultural and Mechanical College has been appointed to an assistant professorship at North Carolina State College.

Mr. H. C. McKenzie of the University of Wisconsin has been appointed to an assistant professorship at Western State College, Gunnison, Colorado.

Professor H. F. MacNeish of Brooklyn College has been appointed to a visiting professorship at the University of Miami.

Mr. C. J. Maloney has been appointed Chief, Statistics Branch, Camp Detrick, Frederick, Maryland.

Mr. B. L. Miller of Swarthmore College has accepted a position as physicist with the Bartol Research Foundation, Franklin Institute.

Mr. B. L. Moyls of Harvard University has been appointed Lecturer at the University of British Columbia.

Dr. E. N. Nilson of Pratt and Whitney Aircraft has accepted a position as analytical engineer at United Aircraft Corporation, East Hartford, Connecticut.

Assistant Professor J. M. H. Olmsted of the University of Minnesota has been promoted to an associate professorship.

Mr. R. R. Reynolds of Oklahoma Agricultural and Mechanical College has been promoted to an assistant professorship.

Dr. L. D. Rodabaugh of the Bureau of the Census has been appointed to an associate professorship at Southern Illinois University.

Mr. Peter Scherk of the University of Saskatchewan has been promoted to an associate professorship.

Professor Joseph Seidlin of Alfred University has been appointed Dean of the Graduate School.

Associate Professor J. Shibli of Pennsylvania State College has been promoted to a professorship.

Mr. E. J. Specht of the University of Minnesota has been appointed to a professorship at Emmanuel Missionary College.

Assistant Professor E. C. Stopher of State Teachers College, Brockport, New York, has been appointed to an assistant professorship at Miami University.

Mr. H. W. Syer of Boston University has been promoted to an assistant professorship.

Mr. John Todd, who is on leave of absence from Kings College, University of London, has been appointed to the staff of the National Bureau of Standards' Institute of Numerical Analysis.

Mrs. John Todd (Dr. Olga Taussky) is a guest worker at the National Bureau of Standards.

Dr. A. H. Van Tuyl of Stanford University has accepted a position as mathematician with the Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland.

Associate Professor Alexander Weinstein of Carnegie Institute of Technology has accepted appointments as senior associate at the Naval Ordnance Laboratory and professor at the University of Maryland.

Assistant Professor J. J. Wheeler of the University of Kansas has retired with the title of Associate Professor Emeritus.

The following appointments to instructorships are announced:

Bethany College: Miss Catherine E. Moser

University of Chicago: Mr. I. R. Hershner

Hunter College: Mr. M. E. White

Johns Hopkins University: Mr. George Shapiro, Mr. D. R. Waterman

Oregon State College: Miss Ella Mae Sowder

University of Texas: Mr. W. L. Shepherd

Mr. James W. Dappert, a civil engineer of Taylorville, Illinois and a charter member of the Association, died December 10, 1947 at the age of eighty-eight years.

Assistant Professor U. P. Davis of the University of Florida died February 9, 1947.

Professor E. E. DeCou of the University of Oregon died October 15, 1947. He was a charter member of the Association.

Professor J. C. Fitterer of Colorado School of Mines died on March 12, 1947. He was a charter member of the Association.

Professor Emeritus William Gillespie of Princeton University died September 13, 1947, at the age of seventy-six years.

Dr. A. M. Harding, formerly president and professor of mathematics and astronomy of the University of Arkansas, died December 24, 1947.

Emeritus Professor G. H. Hardy of Cambridge University, Cambridge, England died December 1, 1947. He was the winner of the Chauvenet Prize for the period 1929–1931.

Dr. A. N. Whitehead, professor of philosophy emeritus of Harvard University, died December 30, 1947.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-FIRST ANNUAL MEETING OF THE ASSOCIATION

The thirty-first annual meeting of the Mathematical Association of America was held at the University of Georgia, Athens, Georgia, on Thursday, January 1, 1948, in conjunction with the annual meeting of the American Mathematical Society. About three hundred and thirty-six persons attended the meetings, including the following one hundred and ninety members of the Association:

Louise Adams, High Point College
V. W. Adkisson, University of Arkansas
R. P. Agnew, Cornell University
G. E. Albert, University of Tennessee
C. B. Allendoerfer, Haverford College
E. A. Bailey, La Grange College
N. H. Ball, U. S. Naval Academy
J. C. Barnes, North Georgia College
C. F. Barr, University of Wyoming
D. F. Barrow, University of Georgia
H. M. Beatty, Ohio State University
E. F. Beckenbach, U.C.L.A.
W. S. Beckwith, University of Georgia
E. G. Begle, Yale University
T. A. Bickerstaff, University of Mississippi
R. C. Blackwell, Furman University
R. G. Blake, University of Florida
L. M. Blumenthal, University of Missouri
Stanley Bolks, Purdue University
M. G. Boyce, Vanderbilt University
J. W. Bradshaw, University of Michigan
A. T. Brauer, University of North Carolina
H. E. Bray, Rice Institute
Foster Brooks, Kent State University
J. W. Brown, Clemson College
N. R. Bryan, University of Maine
R. S. Burington, Bureau of Ordnance, Navy Department
L. P. Burton, University of Alabama
L. E. Bush, College of St. Thomas
Margaret C. Byrne, St. Joseph's College for Women
S. S. Cairns, Syracuse University
Iris Callaway, University of Georgia
E. A. Cameron, University of North Carolina
C. C. Camp, University of Nebraska
Virginia Carlton, Centenary College
W. B. Carver, Cornell University
Randolph Church, U. S. Naval Postgraduate School
R. V. Churchill, University of Michigan
F. E. Clark, Duke University
A. C. Cohen, Jr., University of Georgia
J. B. Coleman, Presbyterian College
Lennie P. Copeland, Wellesley College
N. A. Court, University of Oklahoma
W. H. H. Cowles, Pratt Institute
J. C. Currie, Louisiana State University
H. B. Curry, Pennsylvania State College
J. H. Curtiss, National Bureau of Standards
C. H. Denbow, Naval Postgraduate School
A. H. Diamond, Oklahoma A & M
R. D. Doner, Alabama Polytechnic Institute
Nelle C. Douglas, University of South Carolina
H. H. Downing, University of Kentucky
W. L. Duren, Jr., Tulane University
L. A. Dye, The Citadel
E. D. Eaves, University of Tennessee
F. A. Ficken, University of Tennessee
Floyd Field, Georgia Tech.
L. R. Ford, Illinois Institute of Technology
Tomlinson Fort, University of Georgia
J. S. Frame, Michigan State College
Gordon Fuller, Alabama Polytechnic Institute
H. K. Fulmer, Georgia Tech
W. A. Gager, University of Florida
H. M. Gehman, University of Buffalo
M. E. Gillis, Blue Mountain College
J. W. Givens, University of Tennessee
Mary A. Goins, Marshall College
J. S. Gold, Bucknell University
Michael Goldberg, Bureau of Ordnance, Navy Department
M. O. Gonzalez, University of Alabama
S. T. Gormsen, University of Florida
W. H. Gottschalk, Institute for Advanced Study
S. H. Gould, Purdue University
Walter Graham, Vanderbilt University
Dorothy Green, Huntingdon College

R. E. GREENWOOD, University of Texas
W. C. GRIFFITH, Centenary College
W. S. GUSTIN, Indiana University
D. W. HALL, University of Maryland
E. A. HEDBERG, University of South Carolina
G. A. HEDLUND, University of Virginia
R. E. HENRY, Newark College, Rutgers University
G. W. HESS, Howard College
E. H. C. HILDEBRANDT, Northwestern University
T. H. HILDEBRANDT, University of Michigan
P. R. HILL, JR., University of Georgia
EINAR HILLE, Yale University
C. H. HOLTON, Georgia Tech.
C. W. HOOK, Georgia Tech.
G. B. HUFF, University of Georgia
ELAINE HUNDERTMARK, Florida State University
W. R. HUTCHERSON, Berea College
L. C. HUTCHINSON, Polytechnic Institute of Brooklyn
R. O. HUTCHINSON, Tennessee Polytechnic Institute
E. D. JENKINS, Kent State University
W. L. JOHNSON, Mississippi Southern College
K. R. JONES, NEPA Project, Oak Ridge, Tennessee
L. O. JONES, William Jewell College
H. T. KARNES, Louisiana State University
A. J. KEMPNER, University of Colorado
J. R. KLINE, University of Pennsylvania
F. W. KOKOMOOR, University of Florida
H. L. KRALL, Pennsylvania State
A. E. LAMPEN, Hope College
A. E. LANDRY, Catholic University of America
G. B. LANG, University of Florida
GILLIE A. LAREW, Randolph-Macon Woman's College
C. G. LATIMER, Emory University
H. L. LEE, University of Tennessee
SOLOMON LEFSCHETZ, Princeton University
R. J. LEVIT, University of Georgia
D. C. LEWIS, University of Maryland
F. A. LEWIS, University of Alabama
G. H. LUNDBERG, Vanderbilt University
R. A. LYTLE, University of South Carolina
C. C. MACDUFFEE, University of Wisconsin
E. L. MACKIE, University of North Carolina
A. C. MADDOX, Northwestern State College
J. D. MANCILL, University of Alabama
G. E. MARKLE, University of Detroit
W. A. MARTIN, Georgia Tech
W. T. MARTIN, Massachusetts Institute of Technology
W. L. MASSEY, University of Chattanooga
ELNA B. MCBRIDE, Memphis State College
DOROTHY MCCOY, Belhaven College
S. W. MCINNIS, University of Florida
E. J. MCSHANE, University of Virginia
L. E. MEHLENBACHER, University of Detroit
C. E. MELVILLE, Clark University
B. E. MESERVE, University of Illinois
H. A. MEYER, University of Florida
D. D. MILLER, University of Tennessee
D. C. MOORE, Emory University
T. W. MOORE, U. S. Naval Academy
W. B. MOYE, Georgia Teachers College
SARA L. NELSON, Georgia State College for Women
P. F. NEMENYI, Naval Ordnance Laboratory
C. V. NEWSOM, Oberlin College
MABEL I. NOWLAN, Trailways Bus
E. B. OGDEN, Union College
MORRIS OSTROFSKY, Duquesne University
E. R. OTT, Rutgers University
F. W. OWENS, Pennsylvania State College
Helen B. OWENS, Pennsylvania State College
W. V. PARKER, University of Georgia
I. E. PERLIN, Georgia Tech.
R. B. PLYMALE, Mercer University
W. G. POLLARD, Oak Ridge Institute of Nuclear Studies
G. B. PRICE, University of Kansas
J. F. RANDOLPH, Oberlin College
ELLEN RASOR, University of South Carolina
ADRIENNE S. RAYL, University of Alabama
L. M. REAGAN, University of Wichita
A. W. RECHT, University of Denver
MINA REES, Office of Naval Research
P. K. REES, Louisiana State University
G. E. REVES, The Citadel
P. R. RIDER, Washington University
R. F. RINEHART, Case Inst. of Technology
J. H. ROBERTS, Duke University
H. A. ROBINSON, Agnes Scott College
L. V. ROBINSON, University of South Carolina
W. J. ROBINSON, Centre College (Kentucky)
CAROL S. SCOTT, St. Petersburg Junior College
P. C. SCOTT, East Carolina Teachers College
W. E. SEWELL, U. S. Army
D. C. SHELDON, Clemson College
T. M. SIMPSON, University of Florida
H. L. SMITH, Louisiana State University
E. L. STANLEY, Clemson College
E. P. STARKE, Rutgers University

A. L. STARRETT, Georgia School of Technology
R. P. STEPHENS, University of Georgia
RUTH W. STOKES, Syracuse University
J. L. SYNGE, Carnegie Institute of Technology
OTTO SZÁSZ, University of Cincinnati
J. S. TAYLOR, University of Pittsburgh
H. P. THIELMAN, Iowa State College
J. M. THOMAS, Duke University
JOHN TODD, National Bureau of Standards
H. C. TRIMBLE, Florida State University
HENRY VAN ENGEN, Iowa State Teachers
T. L. WADE, Florida State University
A. D. WALLACE, Tulane University
J. A. WARD, University of Georgia
BETTY R. WEBER, University of South Carolina
K. W. WEGNER, Carleton College
G. T. WHYBURN, University of Virginia
W. L. WILLIAMS, University of South Carolina
R. L. WILSON, University of Tennessee

Rooms for members of the organizations and their families were provided in three of the University dormitories and meals were served in the cafeteria in Snelling Hall.

On Monday, members of the University faculty conducted tours of antebellum homes in Athens. Tea was served in Lyndon Hall on Tuesday afternoon by the ladies of the Department of Mathematics of the University of Georgia. On Tuesday evening a program of piano and violin music was presented in the University Chapel by Miss Despy Karlas and Mr. Robert Harrison, of the University's Department of Music. On Wednesday, the Department of Fine Arts offered an Art Exhibit in the Fine Arts Building, featuring selections from the paintings of Mr. Lamar Dodd and the collection of American paintings donated to the University of Georgia by Mr. Alfred Holbrook.

A dinner for members of the two organizations was held at 8 p.m. on Wednesday in Snelling Hall. Professor Tomlinson Fort, acting as toastmaster, sketched the history of Mathematics at the University of Georgia, and introduced the members of its Department of Mathematics. President Harmon W. Caldwell of the University welcomed the visiting organizations and spoke of the recent expansion of the University. After vocal solos by James Griffith, accompanied by Byron Walker, talks on Applied Mathematics were given by Professor J. L. Synge, Dr. J. H. Curtiss, and Professor H. B. Phillips. An amusing skit, entitled "Applied Mathematics" was presented by the members of the Department of Mathematics and their families. Professor Fort modestly appeared in answer to shouts of "Author! Author!" Group singing led by Professor J. S. Frame ushered in the New Year with the singing of the traditional *Auld Lang Syne*.

At the dinner resolutions were presented by Professor R. P. Agnew and adopted by a rising vote expressing profound appreciation to the members of the University of Georgia for their gracious hospitality.

The sessions of the American Mathematical Society were held from Monday to Wednesday, December 29–31, 1947. The twenty-first Josiah Willard Gibbs Lecture was delivered by Professor P. M. Morse of the Massachusetts Institute of Technology on Monday evening, the title being "Mathematical problems in operations research." On Tuesday afternoon, Professor E. F. Beckenbach, of the University of California at Los Angeles, gave an address by invitation on "Convex functions." Professor T. H. Hildebrandt of the University of Michigan

gave the retiring presidential address on Wednesday morning on the subject: "Integration in abstract spaces."

The Mathematical Association held its sessions on Thursday morning and afternoon in the auditorium of Connor Hall. The program was arranged by the following committee: Walter Leighton, Chairman, Herbert Busemann, and J. A. Cooley.

First Session of the Association

Retiring presidential address: "The scholar in a scientific world," by Professor C. C. MacDuffee, University of Wisconsin.

"On the moduli of the roots of the derivative of a polynomial," by Professor H. E. Bray, The Rice Institute.

"Some applications of topology," by Professor P. A. Smith, Columbia University.

Second Session of the Association

"Opportunities for advanced study offered by the Oak Ridge Institute of Nuclear Studies," by Dr. W. G. Pollard, Oak Ridge Institute of Nuclear Studies.

"The Mathematician in civil service," by Dr. J. H. Curtiss, National Bureau of Standards.

Symposium: "College entrance requirements in mathematics." Moderator: Professor C. V. Newsom, Oberlin College.

(a) "The problem from the point of view of the university teacher," by Professor A. J. Kempner, University of Colorado.

(b) "The problem from the point of view of the secondary school teacher," by Professor Carl N. Shuster, State Teachers College, Trenton, New Jersey.

Meeting of the Board of Governors

The Board met on Wednesday at 2:30 p.m. in Lyndon Hall. Seventeen members of the Board were present. Among the more important items of business transacted were the following.

The one hundred and twenty-two persons listed below, were elected to membership on applications duly certified.

Jane Ammerman, B.S. (Duke Univ.) Instr., Newark Coll. of Rutgers Univ., Newark, N. J.

H. M. Anderson, Ph.M. (Univ. of Wis.) Asst. Prof., Gustavus Adolphus Coll., St. Peter, Minn.

Fay Aufenberg, B.S. in Engg. (Univ. of Mich.) Asst., Univ. of Ill., Navy Pier, Chicago, Ill.

W. G. Bade, B.S. (Calif. Inst. of Tech.) Teaching Asst., Univ. of Calif. at Los Angeles, Los Angeles, Calif.

H. H. Barnett, A.B. (Univ. of Kansas) Asst. Instr., Univ. of Kansas, Lawrence, Kans.

Winifred V. Berglund, M.A. (Northwestern Univ.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

Col. W. W. Bessell, Jr., C.E. (Rensselaer Poly. Inst.) Prof., U. S. Military Acad., West Point, N. Y.

Laura Blakeley, MA. (George Peabody Coll.) Teacher, Dayton High School, Dayton, Ky.

C. F. BRUMFIEL, S.M. (Univ. of Chicago) 306 N. Talley St., Muncie, Ind.

J. A. CARPENTER, A.B. (Univ. of North Carolina) Actuarial Clerk, Pilot Life Insurance Co., Greensboro, N. C.

SISTER M. ROSE AGNES CAVANAGH, M.A. (Catholic Univ. of America) Instr., Salve Regina Coll., Newport, R. I.

G. B. CHARLESWORTH, M.A. (Cambridge Univ.) Instr., Hofstra Coll., Hempstead, N. Y.

B. B. CLARK, B.S. (Univ. of New Mexico) Grad. Student, Oberlin Coll., Oberlin, Ohio

J. J. CLARK, M.A. (New York Univ.) Instr., Adelphi Coll., Garden City, N. Y.

MARGARET L. COMSTOCK, B.A. (Marygrove Coll.) Instr., Univ. of Detroit, Detroit, Mich.

V. F. COWLING, Ph.D. (Rice Inst.) Asst. Prof., Lehigh Univ., Bethlehem, Pa.

MARJORIE L. CROFT, M.A. (Loyola Univ.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

RUTH L. DAVIDS, B.Sc. (New Jersey Coll. for Women) Instr., Newark Coll. of Rutgers Univ., Newark, N. J.

H. F. DEFRANCESCO, B.E.E. (Univ. of Va.) Grad. Student, Univ. of Virginia, Charlottesville, Va.

MAX EPSELL, B.S. (Coll. of the City of New York) 698 Ashford St., Brooklyn 7, N. Y.

P. L. EVANS, M.S. (Kansas State Coll.) Asst. Prof., Kent State Univ., Kent. Ohio

D. J. EWY, A.B. (Univ. of Calif). Instr., Bethel Coll., North Newton, Kans.

E. E. FLOYD, B.A. in Educ. (Univ. of Ala.) Student, Univ. of Virginia, Charlottesville, Va.

KATHERINE S. FOOTE, M.S. (Louisiana State Univ.) Critic Teacher, Miss. Southern Coll., Hattiesburg, Miss.

J. W. GADDUM, M.A. (Univ. of Mo.) Instr., Univ. of Mo., Columbia, Mo.

E. A. GOODHUE, M.S. (Mo. School of Mines) Asso. Prof., Missouri School of Mines, Rolla, Mo.

DOROTHY E. GREEN, M.A. (George Peabody Coll.) Chm. of Math. Dept., Huntingdon Coll., Montgomery, Ala.

R. L. GREENE, B.E.E. (Clarkson Coll.) Instr., Clarkson Coll., Potsdam, N. Y.

MADELEINE GRENARD, A.M. (Univ. of Nebr.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

K. L. HANKERSON, M.S. in Educ. (Univ. of North Dakota) Asst. Prof., Univ. of N. D., Grand Forks, N. D.

DOROTHEA HARDER, M.A. (Univ. of Wash.) Instr., State Coll. of Wash.,. Pullman, Wash.

H. J. HARRJE, Architecture (Cooper Union) Civil Eng. (Coll. of the City of New York) P.O. Box 746, Jacksonville Beach, Fla.

R. E. HENRY, M.Sc. (New York Univ.) Asso. Prof., Newark Coll. of Rutgers Univ., New, ark, N. J.

ROSE L. HORNACEK, M.A. (Loyola Univ.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

HELEN P. HOFFMAN, M.S. (Univ. of Ill.) Instr., Yankton Coll., Yankton, S. D.

R. A. HUESTON, JR., M.A. (Brown) Instr., Chauncy Hall School, Boston, Mass.

BURROWES HUNT, A.B. (Princeton) Grad. Student, Univ. of Colorado, Boulder, Colo.

K. R. JONES, M.S. (Univ. of Mich.) Asso. Physicist, NEPA Project, Oak Ridge, Tenn.

J. A. KERALLA, B.S. (Coll. of William & Mary) Research Div., Elec. Storage Battery Co. (Exide), 1154 Sanger St., Phila., Pa.

E. G. KIMME, Student, Calif. Inst. of Tech., Pasadena, Calif.

E. H. KINGSLEY, B.S. (Central YMCA Coll.) Lecturer, Roosevelt Coll., Chicago, Ill.

CELIA E. KLOTZ, M.A. (Nebr. State Univ.) Instr., Wash. State Coll., Pullman, Wash.

S. G. KNEALE, A.B. (Univ. of Kansas) Asst. Instr., Univ. of Kansas, Lawrence, Kans.

DAVID KOTLER, M.A. (Columbia Univ.) Asst. Prof., Mohawk Coll., Utica 5, N. Y.

OBERT KRUGER, B.A. (Univ. of Nebr.) Prof., St. John's Coll., Winfield, Kans.

ROSE LARIVIERE, B.A. (McGill Univ.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

ORA F. LATHAM, M.A. (Univ. of Ill.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

B. E. LAWRENCE, B.S. (Calif. Inst. of Tech.) Supervisor, Analysis Group, USNOTS, Inyokern, Calif.

H. S. LEVIN, S.B. (Univ. of Chicago) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

JOSEPH LEVITT, B.M.E. (Cooper Union) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

GENE LEVY, M.A. (Univ. of Okla.) Chm. of Math. Dept., College of Ozarks, Clarksville, Ark.

SISTER MARY TERESINE LEWIS, Ph.D. (Catholic Univ. of America) Instr., Fontbonne Coll., St. Louis 5, Mo.

MARGARET LIECHTY, B.A. (Oberlin Coll.) Instr., Penn. State Undergrad. Center, Hazleton, Pa.

WALTER LYCHE, M.A. (Univ. of Minn.) Hd. of Math. Dept., Wartburg Coll., Waverly, Iowa

G. E. MARKLE, M.Sc. (Univ. of Detroit), M.A. (Univ. of Mich.) Asst. Prof., Univ. of Detroit, Detroit, Mich.

MARGARET E. MARTINSON, A.B. (Washburn Municipal Univ.) Instr., Washburn Univ., Topeka, Kans.

R. H. MASON, A.M. (Syracuse Univ.) Instr., Univ. of Fla., Gainesville, Fla.

D. W. MATLACK, B.A. (Grinnell Coll.) Mathematician, North American Aviation, Inc., 3008-B 3rd St., Santa Monica, Calif.

JUNE M. MCARTNEY, B.A. (D'Youville Coll.) Instr., Univ. of Buffalo, Buffalo, N. Y.

W. H. MCBRIDE, M.S. (Univ. of North Dakota) Instr., Univ. of N. D., Grand Forks, N. D.

J. A. MCGRAIL, B.S. (Mich. Coll. of M. & T.) Instr., Univ. of Detroit, Detroit, Mich.

J. J. MCLAUGHLIN, M.A. (Univ. of Mich.) Chm. of Dept., Wisconsin State Teachers Coll., River Falls, Wis.

ELAINE E. MEYER, B.A. (Huron Coll.) Hd. of Math. Dept., Vermillion Public High School, Vermillion, S. D.

E. W. MONTROLL, Ph.D. (Univ. of Pittsburgh) Asst. Prof., Univ. of Pittsburgh, Pittsburgh 13, Pa.

R. C. MOUREAU, B.Sc. (U. S. Naval Acad.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

NICHOLAS MUSSELMAN, M.A. (Mich. State) Instr., Michigan State College, East Lansing, Mich.

W. H. NEELY, M.S. (Univ. of Southern Calif.) Asst. Prof., Texas Christian Univ., Fort Worth, Tex.

T. A. NELSON, M.Ed. (Univ. of Maryland) Instr., Muhlenberg Coll., Allentown, Pa.

GRACE M. NOLAN, M.A. (Loyola Univ.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

R. W. OESCH, M.A. (Univ. of Ill.) Asst. Prof., Trinity Univ., San Antonio, Tex.

F. R. OHNSORG, B.A. (St. Thomas Coll.) Student, Univ. of Minn., St. Paul, Minn.

C. E. OLSEN, B.S. (U. S. Naval Acad.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

C. L. OLSON, Student, George Pepperdine Coll., Los Angeles, Calif.

T. B. ONDRAK, B.A. (St. Procopius Coll.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

HYMAN ORLIN, B.B.A. (Coll. of the City of New York) Jr. Mathematician, U. S. Coast & Geodetic Survey, Washington, D. C.

MORRIS OSTROFSKY, Ph.D. (Univ. of Wis.) Prof., Duquesne Univ., Pittsburgh, Pa.

C. G. PECKHAM, M.S. (Univ. of Ill.) Asst. Prof., Univ. of Dayton, Dayton, Ohio

CAPT. WILLIAM PENNINGTON, AC, 928 Johns Rd., Augusta, Ga..

J. C. PETERSON, M.S. in Ed. (Univ. of North Dakota) Instr., Univ. of N. D., Grand Forks, N. D.

MARY PETTUS, M.A. (Univ. of Chicago) Prof., Lander Coll., Greenwood, S. C.

MARGARET M. PIHLBLAD, A.B. (Kansas Univ.) Grad. Student, Kansas Univ., Lawrence, Kans.

D. W. POUNDER, B.A. (Univ. of Toronto) Grad. Asst., Ill. Inst. of Tech., Chicago, Ill.

BEULAH PROTSMAN, Community High School, Blue Island, Ill.

R. M. RANKIN, M.A. (Univ. of Chicago) Prof. Mo. School of Mines, Rolla, Mo.

J. G. RENNO, M.S. (Mich. State Coll.) Grad. Asst., Univ. of Wis., Madison, Wis.

J. D. RILEY, A.B. (Park Coll.) Asst. Instr., Univ. of Kansas, Lawrence, Kans.

P. A. ROGNLIE, M.S. (Univ. of N. D.) Asso. Prof., Univ. of N. D., Grand Forks, N. D.

HANAN RUBIN, A.B. (New York Univ.) Instr., New York Univ., New York, N. Y.

H. J. RYSER, M.A. (Wisconsin) Univ. of Wis., Madison, Wis.

A. H. SARNO, M.S. (Coll. of the City of New York) Instr., St. John's Univ., Brooklyn, N. Y.

ROBERT SCHATTEN, Ph.D. (Columbia Univ.) Asso. Prof., Univ. of Kansas, Lawrence, Kans.

CECIL SCHWARTZ, B.S. (Miss. State Coll.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

CAROL S. SCOTT, M.A. (Western Reserve Univ.) St. Petersburg Jr. Coll., St. Petersburg, Fla.

HELEN W. SEARS, M.A. (Loyola Univ.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

C. F. SEBASTA, M.A. (Univ. of Pittsburgh) Instr., Univ. of Pittsburgh, Ellsworth Center, Pittsburgh, Pa.

BERNARD SHERAK, M.A. (Cornell) Instr., Rutgers Univ., Newark Coll., Newark, N. J.

I. W. SILVERSTEN, B.A. (Brooklyn Coll.) Supervisor, Babcock & Wilcox Co., New York, N. Y.

ALBERT SOGLIN, B.E. (Chicago Teachers Coll.) Student, Ill. Inst. of Tech., Chicago, Ill.

HERBERT SOLOMON, M.A. (Columbia Univ.) Special Lecturer, Newark Coll. of Engg., Newark, N. J.

FLORENCE N. SOLOMONT, B.A. (Radcliffe) 312 Hammond Pond Parkway, Chestnut Hill. 67, Mass.

G. A. SPOONER, B.S. (Teachers Coll. of Conn.) Asst. Instr., Teachers Coll., New Britain, Conn.

P. C. SQUIRES, Ph.D. (Princeton Univ.) Asst. Prof., Sampson Coll., Sampson, N. Y.

LOIS B. STELLING, B.S. (Univ. of Chicago) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

R. G. STONEHAM, Sc.M. (Brown Univ.) Teaching Asst., Univ. of Calif., Berkeley, Calif.

W. B. STOVALL, JR., B.S. (Univ. of Fla.) Grad. Fellow, Univ. of Fla., Gainesville, Fla.

FLORENCE SWALLOW, M.A. (Univ. of Ill.) Instr., North Dakota Agric. Coll., Fargo, N. D.

D. T. TEODORO, B.S. (Univ. of Mich.) Instr., Univ. of Detroit, Detroit, Mich.

L. O. THOMPSON, B.S. (New River State Coll.) Instr., Univ. of Detroit, Detroit, Mich.

D. L. THOMSEN, JR., Ph.D. (Mass. Inst. of Tech.) Instr., Haverford Coll., Haverford, Pa.

R. R. TOWNSEND, B.S. (Muhlenberg Coll.) Instr., Penn. State Coll., Schuylkill Undergrad. Center, Pottsville, Pa.

A. T. STREET, M.A. (Northwestern Univ.) Asso. Prof., Roosevelt Coll., Chicago, Ill.

I. F. WAGNER, JR., M.S. (Virginia Poly Inst.) Instr., Univ. of Va., Charlottesville, Va.

F. A. WALLACE, M.S. (Univ. of Florida) Instr., Jacksonville Jr. Coll., Jacksonville, Fla.

E. H. WANG, Diploma Engineer, Techn. (Univ., Vienna) Instr., Univ. of Cincinnati, Cincinnati, Ohio

MARION L. WEST, Student, Univ. of Tulsa, Tulsa, Okla.

CHRISTINE WESTGATE, M.S. (Univ. of Chicago) Instr., Univ. of N. D., Grafton, N. D.

M. E. WHITE, B.A. (Wesleyan Univ.) Instr., Hunter Coll., New York City, N. Y.

R. M. WHITMORE, M.A. (Univ. of Texas) Asst. Prof., Southwestern Univ., Georgetown, Tex.

G. T. WILLIAMS, Brookhaven National Lab., Upton, N. Y.

C. C. WILSON, B.Ed. (Chicago Teachers Coll.) Instr., Univ. of Ill., Navy Pier, Chicago, Ill.

J. L. ZEMMER, JR., M.S. (Tulane Univ.) Grad. Asst., Univ. of Wis., Madison, Wis.

L. J. ZIMMERMAN, B.A. (Goshen Coll.) Instr., Goshen Coll., Goshen, Ind.

The Secretary reported the death of the following members of the Association:

H. M. ACKLEY, Professor, Western Michigan College. (February 9, 1947)
H. G. AVERS, Chief Mathematician, U. S. Coast and Geodetic Survey. (January 19, 1947)
ETTORE BORTOLOTTI, Professor, University of Bologna, Italy. (February 17, 1947)
J. F. BUTLER, Associate Professor, University of Detroit. (February 18, 1947)
C. C. CARTER, Attorney, Bluffs, Illinois. (May 5, 1947)
J. W. DAPPERT, Civil Engineer, Taylorville, Illinois. (December 10, 1947)
E. E. DECOU, Professor Emeritus, University of Oregon. (October 15, 1947)
G. W. EVANS, Retired, Charlestown High School, Charlestown, Massachusetts. (February, 1947)
B. F. FINKEL, Professor Emeritus, Drury College. (February 5, 1947)
J. C. FITTERER, Professor, Colorado School of Mines. (March 12, 1947)
WILLIAM GILLESPIE, Professor Emeritus, Princeton University. (September 13, 1947)
G. H. HARDY, Professor Emeritus, Cambridge University, England. (December 1, 1947)
CORA B. HENNEL, Professor, Indiana University. (June 26, 1947)
BERNARD MASON, Instructor in Physics, Hofstra College. (August 25, 1947)
W. A. MOODY, Professor Emeritus, Bowdoin College. (February 3, 1947)
WILLIAM ORANGE, Professor, Los Angeles City College. (December 9, 1946)

B. L. REMICK, Professor Emeritus, Kansas State College. (March 18, 1947)
W. C. RUFUS, Professor of Astronomy, University of Michigan. (September 21, 1946)
W. T. SHORT, Professor, Oklahoma Baptist University. (February 19, 1947)
S. A. SINGER, Professor, Capital University. (April 25, 1947)
G. L. WINKELMANN, Reverend, St. John's University, Minnesota. (January 23, 1947)

The Board voted to accept the invitations of Ohio State University to hold the Annual Meeting of 1948 at Columbus, Ohio. It was also voted to hold the Annual Meeting of 1949 in New York City.

On nomination by the Executive Committee, the Board voted to elect W. B. Carver a member of the Finance Committee for the four years, 1948–51. The Board approved the appointment by President Ford of a Nominating Committee for 1948 consisting of W. B. Carver, G. C. Evans, and C. C. MacDuffee, Chairman. President Ford also announced the appointment of E. J. McShane as a member of the Council of the American Association for the Advancement of Science for the two-year term 1948–49.

The retiring Secretary-Treasurer, W. B. Carver, was requested to express to the President of Cornell University the thanks of the Association for having contributed office space to the Association for the past five years.

Arrangements were authorized for the printing and distribution of the Eighth Carus Monograph, "Rings," by N. H. McCoy.

On the nomination of the Editor-in-Chief, C. V. Newsom, the Board elected the following Associate Editors of the MONTHLY for the year 1948:

C. B. ALLENDOERFER	H. P. EVANS	L. F. OLLMANN
E. F. BECKENBACH	HOWARD EVES	R. F. RINEHART
L. M. BLUMENTHAL	G. E. HAY	EDITH R. SCHNECKENBURGER
N. B. CONKWRIGHT	N. H. MCCOY	E. P. STARKE
H. S. M. COXETER	W. T. MARTIN	E. P. VANCE

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Thursday at 2:30 p.m., in Connor Hall, with President Ford presiding.

The tellers, H. E. Bray and P. K. Rees, announced the following results of the balloting for officers:

Saunders MacLane, University of Chicago, was elected First Vice-President for the two-years 1948–49.

C. R. Adams, Brown University, and W. L. Ayres, Purdue University, were elected Governors at Large for the three-year term 1948–50.

Announcement was made of the award of the Chauvenet Prize for the years 1944–46 to Professor P. R. Halmos of the Institute for Advanced Study for his paper: "The Foundations of Probability," this MONTHLY, Vol. 51 (1944), pp. 493–510. The Committee on the Chauvenet Prize consisted of R. P. Agnew, chairman, R. W. Barnard, and R. P. Boas, Jr.

On motion of C. G. Latimer and W. L. Duren, the following resloution was unanimously adopted:

Whereas, Professor W. B. Carver has just completed a five-year period as Secretary-Treasurer of the Association, and

Whereas, this was preceded by a two-year term as President and also by a five-year term as Editor-in-Chief of the MONTHLY,

Be it resolved, that the Mathematical Association of America hereby expresses to Professor Carver its profound thanks for the long, devoted, and efficient services which he has so freely given to this Association. His efforts during the difficult periods in which he has held office, have contributed in large measure to the functioning of the Association.

At the conclusion of the symposium on "College entrance requirements in mathematics," a resolution was adopted requesting the Board of Governors to study in cooperation with the National Council of Teachers of Mathematics and with other groups, the problem of the maintenance of standards in the preparation of students for college, including the study of certification of teachers, teacher-training programs, and curricular questions.

H. M. GEHMAN, *Secretary-Treasurer*

MAY MEETING OF THE ILLINOIS SECTION

The twenty-sixth annual meeting of the Illinois Section of the Mathematical Association of America was held at Wheaton College, Wheaton, Illinois, on Friday and Saturday, May 9–10, 1947. Professor C. N. Mills of Normal University presided at all sessions.

There were sixty-seven in attendance including the following thirty-seven members of the Association: D. L. Arenson, H. G. Ayre, J. K. Baumbart, S. F. Bibb, G. M. Bloom, Fanny W. Boyce, B. K. Brown, Laura Christman, E. G. H. Comfort, J. J. Corliss, W. H. Coulter, L. R. Ford, D. M. Friedlen, J. W. Givens, G. D. Gore, E. D. Hellinger, M. R. Hestenes, E. H. C. Hildebrandt, Norbert Kaufman, E. C. Kiefer, R. A. Kliphardt, W. C. Krathwohl, A. T. Lonseth, C. T. McCormick, Karl Menger, C. N. Mills, C. W. Moran, H. E. Nelson, Rufus Oldenberger, Margaret Olmsted, Gordon Pall, H. A. Poppen, W. T. Reid, Haim Reingold, M. Anice Seybold, Malcolm Smith, and L. R. Wilcox.

The officers for 1947–48 were elected as follows: Chairman, J. J. Corliss, Navy Pier, Chicago; Vice-Chairman, W. C. Krathwohl, Illinois Institute of Technology; Secretary, Earl C. Kiefer, Millikin University. The meetings for 1948 are scheduled to be held at the Illinois Institute of Technology, Chicago, on Friday and Saturday, May 8–9, 1948. A resolutions committee composed of L. R. Ford, Clyde McCormick, and Margaret Olmsted was appointed by the Chairman, and reported at the business meeting on Saturday morning.

The following papers were presented:

1. *Some construction problems with quadratic forms*, by Dr. Gordon Pall, Illinois Institute of Technology.

A genus of integral quadratic forms can be characterized by its determinant, index, and the residue of one of its forms modulo $8d$, where d is the determinant and k is the product of the distinct odd primes in the determinant. Conversely, a series of theorems of which the following is one of the

simplest can be proved: Let $n \geqq 1$, d and i be given integers, $(-1)^i d > 0$, $0 \leqq i \leqq n$, and let ρ be a form with an integral matrix such that the determinant of ρ is congruent to $d \pmod{8d^2}$, and such that

$$\prod_{p/2d} c_p(\rho) = (-1)^{(i-1)/2},$$

where q is a non-zero integer. Then we can construct an n-ary integral form f of determinant d and index i, satisfying $f \equiv \rho \pmod{8dkq}$ except that if n is even, ρ is the double of a non-classic form, and if d is odd, then the modulus is $4dkq$. As an immediate application we have a completely elementary proof of the theorem of Siegel that if one form represents another congruencially for every modulus, and in the field of reals, then some form in the same genus as the first represents the second form integrally.

2. *Artin's treatment of the gamma function*, by Margaret S. Matchett, Illinois Institute of Technology, presented by Dr. L. R. Ford.

The Gamma Function can be characterized uniquely by the conditions that $\Gamma(1)=1$, $\Gamma(x+1)=x\Gamma(x)$, and that $\Gamma(x+1)$ is logarithmically convex. The function $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$ is shown to satisfy these conditions. It is then shown that the conditions cited, and the existence of a function satisfying them, imply that

$$\lim_{n\to\infty} \frac{(n-1)!n^x}{x(x+1)\cdots(x+n-1)}$$

exists, and that any function satisfying the conditions is equal to this expression.

3. *The beginning teacher*, by Dr. Haim Reingold, Illinois Institute of Technology.

Due to the increased registration in American universities there is a large increase in the number of students taking mathematics. This has brought about a shortage of qualified instructors, and a large proportion of college mathematics classes throughout the country is being taught by graduate students, high school teachers, and others. Professor Reingold discussed the problems and difficulties which face these beginning teachers.

4. *Elasticity of demand and supply, a general study*, by Dr. Lewis A. Maverick, Southern Illinois Normal University, introduced by Dr. H. G. Ayre.

(The writer distributed to the audience a mimeographed exposition. This he did not read in full, but spoke from it, indicating the nature and significance of the study.)

There are here gathered together, from sundry sources, contributions to one or another aspect of elasticity. An explanation is given of the significance to economists of this concept. Several simple curves are employed to represent demand and supply; and for each the elasticity is investigated. General (non-economic) functions are examined. Functions of constant elasticity are of particular interest, they appear on log-log paper as straight lines; they may be employed to show "the" elasticity (the average elasticity) between two given points, and may profitably be fitted, by least squares, when more than two points are given. An empirical procedure is suggested for drawing a demand curve of "reasonable" or "not improbable" shape through a single point.

5. *Some aspects of the theory of traffic control*, by Dr. L. R. Wilcox, Illinois Institute of Technology.

The mathematical theory of "progressive" periodic traffic light networks on a rectangular lattice is developed. The concept of a network admitting non-stop travel at some speed in both directions along each path is formulated mathematically. The relation thus defined between (a) the geometric pattern of paths and intersections, (b) the behavior of the control lights and (c) the effective speeds along the various paths, is completely determined. Thus all lattices of intersections are found which admit the existence of suitable settings under (b) for which speeds (c) may be

found. For each such lattice, all settings of the lights admitting speeds (c) are determined, and for each such setting all speeds are found.

The condition on (a) is shown to be the commensurability of certain distances. A necessary condition on (b) is the equality of the periods of all the lights. Necessary and sufficient conditions on (b) and (c) are so formulated that either (b) or (c) may be presented within determined limitations and the other found.

The theory of periodic sets of real numbers which are sums of intervals is the basic mathematical tool, since such sets characterize (b) completely. The relations connecting (b) and (c) result from the solution of certain simultaneous congruences.

6. *The Life of Florian Cajori*, by Dr. H. T. Davis, Northwestern University, introduced by Dr. C. N. Mills.

The speaker was in a class under Dr. Cajori when in college. He gave many highlights of Cajori's teaching, as well as an historic biography.

7. *Special cryptoform*, by Mr. W. H. Coulter, Decatur, Illinois.

Mr. Coulter presented problem E-751 with a number of remarks and observations regarding such problems.

8. *Practical computational methods for the solution of equations*, by Dr. Rufus Oldenberger, Illinois Institute of Technology.

The methods of solving ordinary differential equations with constant coefficients, and the corresponding algebraic equations as taught in the colleges and universities, are too tedious and complicated for the engineer who needs to obtain his solution in a few minutes. The author has found that certain simple techniques involving modified synthetic division processes save a tremendous amount of time over well known methods. A typical involved known method is that of Graeffe's for finding the complex roots of an algebraic equation. If we teach simpler methods, the average engineer, who today does not use differential equations, may be encouraged to employ mathematics beyond the calculus, and will find that he can do his work much more efficiently by combining mathematical analysis with experiment. Teachers of mathematics are largely to blame for industry's failure to make more use of it. Instead of teaching many ways of solving many problems, thus confusing the student, we should emphasize the solution of a few types which are sure to occur in all fields of technical work.

9. *Self dual postulates in projective geometry*, by Dr. Karl Menger, Illinois Institute of Technology.

While the geometry of the projective plane satisfies the principle of duality, the traditional postulates which form its foundation are not self-dual. For example, the duals of the postulates that there are three non-collinear points, and that there are at least three points on each line (i.e., the statements that there are three non-current lines and at least three lines on each point) are not among the postulates because they can be derived from them. The following four self-dual independent propositions are equivalent to the traditional foundation of projective geometry: I. *There is at least one line on every two points, and at least one point on every two lines.* II. *No two points are on two lines.* III. *There are two points p_1, p_2 and two lines l_1, l_2 such that p_i is on l_j if and only if $i=j$.* IV. *There are two points, p, q and two lines l, m such that the intersection of l and m is on the join of p and q.* We obtain all the classical projective planes except Fano's seven-point-plane with three points on each line, if we replace III and IV by III*. III*. *There exist three non-collinear points p_1, p_2, p_3 and three non-concurrent lines l_1, l_2, l_3 such that p_i is on l_j if and only if $i=j$.* The laws of Desargues and Pappus are capable of self-dual formulations. Moreover, one can formulate a self-dual foundation of the geometry of the three-dimensional projective space in terms of points, lines, and planes. (Cf. *Reports of a Mathematical Colloquium*, Notre Dame, issue 8, 1947).

E. C. KIEFER, *Secretary*

MAY MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the State Teachers College in St. Cloud, Minnesota, on Saturday, May 10, 1947. Two sessions were held in the forenoon, one at luncheon, and one in the afternoon. Sister M. Seraphim Gibbons, Professors C. O. Bemis, W. L. Hard, and K. H. Bracewell presided at the respective sessions.

Eighty persons attended the meeting, including the following twenty-nine members of the Association: K. H. Bracewell, R. W. Brink, L. E. Bush, W. H. Bussey, R. H. Cameron, E. J. Camp, C. S. Carlson, H. D. Colson, Walter Fleming, Gladys Gibbens, Sister M. Seraphim Gibbons, H. W. Godderz, W. L. Hart, D. A. Johnson, G. K. Kalisch, S. L. Mason, Kenneth May, Sister M. Joanne Muggli, M. J. Norris, J. M. H. Olmsted, Sister Claudette Scoblic, L. W. Sheridan, F. C. Smith, Marion Smith, R. C. Staley, F. J. Taylor, Takashi Terami, Ella Throp, and K. W. Wegner.

The following officers were elected for the coming year: Chairman, H. L.Turrittin, University of Minnesota; Secretary, L. E. Bush, College of St. Thomas; Executive committee, E. J. Camp, Macelaster College; Kenneth May, Carleton College; Charles Hatfield, Jr., University of Minn.

A tribute covering the life, work, and character of Professor Dunham Jackson, who died November 6, 1946, was read by Professor Bussey. A tribute covering the life, work, and character of The Reverend Gilbert L. Winkelmann, who died January 23, 1947, was read by Professor Taylor. Announcement was made of the election of Professor L. E. Bush, College of St. Thomas, as Sectional Governor for the term beginning July 1, 1947.

The Executive Committee was instructed to prepare a set of by-laws for the Section, to be presented for approval at the 1948 meeting of the Section. The Committee was further instructed to consider the advisability of enlarging the Section to include parts of the neighboring States, not now included in any Section of the Association.

By invitation of the Executive Committee, Professor Kenneth May delivered an address at the second morning session. The title of his address was *Probabilities of Certain Election Results.*

The speaker discussed the probability of a minority victory in an indirect or representative election. Given n districts, in each of which m votes were cast for one of the other of the two parties according to uniform independent distributions, it is desired to find the probability that one or the other party win a majority of the districts with a minority of the total popular vote. For small values of m and n, classical enumerative methods suffice. For large values of m and n, results can be obtained by utilizing character functions and Bernstein's central limit theorem. For three districts, the probability was shown to increase from $3/32$ for $m=3$, rapidly approaching $1/8$ with increasing m. Other values and formulas were found to complete the borders of a table of the probability as a function of m and n. With increasing m and n, the probability was shown to approach $1/6$ rapidly.

In addition to the lecture by Professor May, the following five papers were presented:

1. *Vector operations as matrices*, by Mr. Chih-yi Wang, University of Minnesota, introduced by Professor John M. H. Olmsted.

This paper was devoted to the representation of the gradient, diverence, and curl operators by means of the matric operators

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}, \quad \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \quad \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix},$$

respectively. It was shown that this representation permits a simple proof of the invariance of these concepts under a transformation from one rectangular coordinate system to another with the same orientation of axes, and their convenience was demonstrated for establishing familiar identities such as those involving the divergence of the curl, and curl of the curl.

2. *Some aspects of systems of partial differential equations*, Sister Joanne Muggli, The College of Benedict, introduced by Sister Claudette Schoblic.

The solution of the system of partial differential equations $D_i u = f_i(x_i, \cdots, x_n)$, where

$$D_i = \frac{\partial^{i_1+i_2+\cdots+i_n}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}, \qquad (i = 1, 2, \cdots, k)$$

was considered. The monomial m_i was defined by $m_i = x_1^{i_1} \cdots x_n^{i_n}$. By use of the algebra of classes, recursion formulas were obtained to express all the monomials which are multiples of at least one m_i but separated into classes so that there is no overlapping of monomials. These recursion formulas lead to an explicit classification of either principal or parametric derivatives. From the formula for the principal derivatives, the integrability conditions are directly given, and from the formula for the parametric derivatives the arbitrary part of the solution is indicated.

3. *A statistical study of the frequency and intensity of storm centers*, by Professor L. W. Sheridan, The College of St. Thomas.

This was a presentation of the results of a study made as an aid to weather forecasting over regions critical to military operations during World War II. The study was illustrated by slides showing charts marked with the frequency and intensity distributions of centers of low pressure over the area investigated.

4. *Representations of real numbers by sequences of integers*, by Professor M. J. Norris, The College of St. Thomas.

The problem of finding a one-to-one mapping f of the real numbers into the family of infinite sequences of integers, such that given a natural number N and a real number x there exists an $\epsilon > 0$, such that $f(y)$ does not differ from $f(x)$ in the first N positions when y differs from x by less than ϵ was proposed. After imposing the Cartesian product topology on the family of sequences, the question was readily answered in the negative.

5. *Some remarks on Makeham's Law*, by Professor Franklin C. Smith, The College of St. Thomas.

After discussing the general problems of approximating mortality statistics with simple mathe-

matical functions, the author presented three famous approximations, known as Gompertz's law, Makeham's First Law, and Makeham's Second Law. Special attention was given to Makeham's First Law and the importance of this law from a practical point of view was stressed.

6. *Ptolemy's Theorem*, by Professor (Emeritus) W. E. Brooke, University of Minnesota.

A proof was given for the theorem: In any quadrilateral inscribed in a circle, the product of the diagonals is equal to the product of two opposite sides plus the product of the other two opposite sides. The following problem was proposed and solved: To construct the circle through two given points and through the extremities of a diameter of a given circle.

7. *A correlation of elementary methods of solving differential equations*, by Professor Neil Lockwood, Duluth State Teachers College (introduced by the Secretary).

The speaker started with the principle that if any operations, such as taking the derivative and multiplying by a constant, follow the commutative, associative, distributive, and other elementary laws of algebra, then these operations follow all the laws of algebra which depend upon these elementary laws, and the symbols of such operations may be treated as if they were ordinary algebraic quantities. It was shown how, by the methods of operators, a large number of types of solutions of differential equations which heretofore seem to have been treated as unconnected topics or as "lucky guesses" may be developed as a unified whole. Among other things, a simple *a priori* derivation was given for the Euler-d'Alembert solution of linear differential equations with constant coefficients, the method of undetermined coefficients, the method of variation of parameters, and the method of solution by infinite series.

The meeting concluded with a panel discussion on the training of secondary school teachers of mathematics. The members of the panel were Professor Kenneth Wegner, Carleton College, Moderator; Professor C. S. Carlson, St. Olaf College; Professor C. O. Bemis, St. Cloud State Teachers College; Sister M. Seraphim Gibbons, The College of St. Catherine. After brief statements by the members of the panel, the audience engaged in a lively discussion of the problem.

L. E. BUSH, *Secretary*

MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The annual meeting of the Upper New York State Section of the Mathematical Association of America was held at the University of Rochester, Rochester, New York, on Saturday, May 10, 1947. Dean W. H. Durfee of Hobart College presided at the morning session, and Professor C. W. Watkeys of the University of Rochester presided at the afternoon session.

About one hundred persons were present, including the following forty-one members of the Association: R. P. Agnew, R. G. Albert, E. B. Allen, H. T. R. Aude, P. R. Bartram, Dorothy L. Bernstein, William Betz, F. J. H. Burkett, S. S. Cairns, I. S. Carroll, W. B. Carver, Rachel Davison, F. F. Decker, E. J. Downie, Walter H. Durfee, C. W. Foard, A. H. Fox, A. S. Gale, H. M. Gehman, B. H. Gere, N. G. Gunderson, May N. Harwood, L. L. Lowenstein, R. R. R. Luckey, Sister Mary Michael (Maloney), L. L. Merrill, Harriet F. Montague, C. W. Munshower, W. V. Nevins, B. C. Patterson, L. R. Polan, Harry Pollard,

M. A. Scheier, Wladimir Seidel, Joseph Seidlin, Ruth W. Stokes, Mary C. Suffa, A. K. Waltz, J. F. Wardwell, C. W. Watkeys, G. M. Wing.

The following officers were elected: Chairman, D. S. Morse, Union College; Vice-Chairman, E. B. Allen, Rensselaer Polytechnic Institute; Secretary, C. W. Munshower, Colgate University. It was agreed to hold the 1948 meeting at Union College, Schenectady, New York, the 1949 meeting at the University of Buffalo, and the 1950 meeting at Syracuse University.

The following papers were presented:

1. *Electronic digital calculators*, By Dr. R. D. O'Neil, Eastman Kodak Co., introduced by Professor Seidel.

The present era, the speaker said, is witnessing a development of automatic calculators which will have a profound effect upon the evolution of mathematics and its applications. These calculators will relieve the mathematician of the burden of arithmetical computation, and will enable him to solve problems which would be impracticable to handle by any other available means. The principal features of the calculators are extreme speed, accuracy, flexibility, and the ability to handle a complete problem automatically. The designers of these computers have presented the mathematician with new responsibilities. He must develop new computing techniques which are better suited to automatic computation, and he must modify his research methods to take better advantage of the help which the computer offers.

2. *Absolutely and completely monotonic functions*, by Professor Harry Pollard, Cornell University.

A function is *absolutely monotonic* on an interval if it and all its derivatives are non-negative there; for example, $f(x)=\sin^{-1}x$, $(0\leqq x\leqq 1)$. Such a function is necessarily analytic in the interval. A function is *completely monotonic* on an interval if $(-1)^k f^{(k)}(x)\geqq 0$ there; if $f(x)$ is completely monotonic on (a, b), then $f(-x)$ is absolutely monotonic on $(-b, -a)$; for example a^{-x}, $a>0$ are completely monotonic on $(0, \infty)$. A function $f(x)$ is completely monotonic on $(0, \infty)$ if, representable in the form $f(x)=\int_0^\infty e^{-xt}d\alpha(t)$, where $\alpha(t)$ is increasing. The connection with the famous moment problem of Hansdorff was discussed. A function is *completely convex* if $(-1)^k f^{(2k)}(x)\geqq 0$ for all k; the Taylor series for such a function necessarily converges for all x, and hence represents an entire function.

3. *Eccentricity and slope*, by Professor Harriet F. Montague, the University of Buffalo.

The author establishes a correspondence between lines of slope m passing through the origin, and families of conics of eccentricity $e=m$ with axes along the coordinate axes and with centers (in the case of the central conics) at the origin, or vertices (in the case of the parabolas) at the origin. Lines of positive slope are associated with families of conics with foci on the x-axis; lines of negative slope are associated with families of conics with foci on the y-axis. The x-axis is associated with a family of circles with the centers at the origin. For the sake of symmetry, the lines of the slope ± 1 are associated with double families of parabolas. A particular member of any family of conics is determined by a pair of points on the associated line at a fixed distance from the origin. Special attention is given to intersections of the conics associated with lines whose slopes are reciprocals.

4. *Familiarity and understanding in mathematics*, by Professor S. S. Cairns, Syracuse University.

Familiarity, rather than understanding, is cultivated by most of our mathematical instruction in schools and colleges. Even the best students generally learn to do problems without knowing why the methods work; for technical skill is easier to teach, easier to grade and more useful in

outside applications than is the underlying theory. In the case of arithmetic, it is necessary to teach technique to children who are too young to appreciate theoretic considerations. The decimal system of notation is regarded by most people as something established by divine decree, rather than as a result of biological accident. With only the most elementary algebraic concepts, one can approach an understanding of arithmetic through a study of the four fundamental operations as they might be developed by creatures with eight or twelve (and so on) fingers instead of ten. One can then proceed to justify these techniques in terms of a general base. Other arithmetical devices, similarly studied, lead to amusing elementary consequences.

5. *The present mathematical situation and next steps in the teaching of mathematics*, by Dr. William Betz, Rochester, N. Y.

The speaker stated that the mathematical scene is still characterized by a confusion of aims, of content, and of organization. The fundamental thesis of the educator is that "academic" mathematics of the usual type is largely "non-functional," and that it does not "meet the needs" of the vast majority of our young people. The real cause of the present breakdown must be located in a widespread educational opportunism which is without an adequate philosophy in dealing with the problems of mass education. Among the most obvious steps needed to correct the present situation are the following: a nation-wide publicity campaign exposing the folly and danger of current policies; a clear and authoritative analysis of the real meaning of "functional competence in mathematics"; the general adoption of a "two-track program" in secondary mathematics; the planning of a continuous curriculum in secondary mathematics, which shall give due attention to understanding, significant application, and mastery.

6. *Panel discussion*: *Some problems relating to mathematics in New York Colleges*, by Professor Wladimir Seidel, the University of Rochester, Chairman; Professor J. M. Synnerdahl, Canisius College; Professor K. E. Bush, Mohawk College; Professor D. E. Kibbey, Syracuse University.

Several problems of the urban college were enumerated by the first speaker. Mr. Bush described several of the courses of instruction in mathematics in the Associated Colleges of Upper New York. He stated that engineering seemed to be the preferred field, and that about three-quarters of the students were studying mathematics. He outlined a number of problems which may arise when the students transfer to other institutions. Dr. Kibbey pointed out that the Associated Colleges and the branch colleges of Syracuse University will feed more students into the advanced courses of the established institutions, and increase the present demand for competent instructors; he predicted enlarged enrollments in such courses as advanced calculus, and raised the question of whether these courses meet the needs of students. Finally, Professor Kibbey discussed the possibility of teaching beginning students some of the fundamental concepts of mathematics to give them some sense of the structure of the science. A general discussion followed.

C. W. MUNSHOWER, *Secretary*

MAY MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the Mathematical Association of America was held at Purdue University, Lafayette, Indiana on May 16–17, 1947. On Friday evening sixty members and guests attended a dinner in honor of President and Mrs. L. R. Ford. After the dinner Professor Ford gave a lecture entitled *Some Remarkable Theorems About Areas*.

Eighty-one persons attended the meetings, including the following thirty members of the Association: W. L. Ayres, L. G. Black, Stanley Bolks, I. W. Burr, G. E. Carscallen, K. W. Crain, P. D. Edwards, L. R. Ford, E. L. Godfrey,

Michael Golomb, G. H. Graves, W. R. Hardman, C. T. Hazard, H. K. Hughes, Rufus Isaacs, H. F. S. Jonah, M. W. Keller, E. L. Klinger, M. M. Lemme, F. C. Leone, G. T. Miller, P. M. Nastucoff, Ivan Niven, P. M. Pepper, J. C. Polley, C. K. Robbins, M. E. Shanks, L. S. Shively, R. B. Stone, M. S. Webster, K. P. Williams.

Professor G. H. Graves, Chairman, presided at the business meeting. Professor P. M. Pepper, Notre Dame, was elected Secretary-Treasurer to succeed Professor M. W. Keller who resigned after serving for six years. It was decided to hold a fall meeting again on October 17, 1947 at Ball State Teachers College, Muncie, Indiana, in conjunction with the fall meeting of the Indiana Academy of Science.

The following papers were presented:

1. *Geometries and their terminology*, by Sister Gertrude Marie, O.S.F., Marian College.

In this discussion the speaker traced in parallel the evolution and naming of the various types of geometry, and the history and significance of the names. The purpose of the study was to contribute to an evaluation of current geometric terminology on scientific and linguistic grounds.

2. *A five significant figure slide rule for plane and spherical trigonometry*, by Professor P. M. Pepper, University of Notre Dame.

Professor Pepper demonstrated a slide rule for performing the computations of plane and spherical trigonometry with an accuracy comparable to that obtained by using five-place logarithmic tables. Besides performing all the usual operations, the speaker illustrated how the rule could be used as a five place table of reciprocals of numbers, of natural trigonometric functions, of logarithms of numbers, and of logarithms of the trigonometric functions.

3. *A simple proof that π is irrational*, by Professor Ivan Niven, Purdue University.

The results obtained in this paper were published in *The Bulletin of the American Mathematical Society*, vol. 53, p. 509.

4. *Remarks on the construction of tables of functions*, by R. D. Gordon, Indiana University, introduced by Professor K. P. Williams.

The speaker discussed briefly the principal devices which make possible the constuction of finite tables so as to accommodate "infinitely many" possible calculations. He also indicated how this subject could be introduced in elementary courses.

5. *A congruence on the sums of powers*, by Gordon Overholtzer, Indiana University, introduced by Professor M. W. Keller.

The methods of investigation of the Schur derivate of a sequence were applied to the summation of the kth powers (k being any integer) of the integers from 1 to p^n (p an odd prime) in a single residue class modulo p.

6. *Functional iterates of half-order*, by Professor Rufus Isaacs, University of Notre Dame.

Let g be a functional mapping any set E into itself. Professor Isaacs discussed the existence of a function f of similar type such that for all x in E, one obtains $f(f(x))=g(x)$. The speaker showed

that the existence criterion is that each linkage can be classified either into a matable pair or as self-matable.

7. *Some consequences of Sterling's formula for log* $\Gamma(z)$, by Professor H. K. Hughes, Purdue University.

In this paper, the speaker derived some series developments which he had occasion to use. The function

$$\frac{\Gamma(\alpha z+a)\Gamma(\beta z+b)}{\Gamma(\gamma z+c)\Gamma(\delta z+d)}$$

is typical of those developed. Here α, β, γ, δ are positive and such that $\alpha+\beta=\gamma+\delta$, and a, b, c, d are any real or complex numbers such that $a+b=c+d$. This function was expanded in the form of a factorial series multiplied by an exponential factor. The results obtained follow from Sterling's formula for log $\Gamma(z)$.

M. W. KELLER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Summer Meeting, Madison, Wisconsin, September 6–7, 1948.

Thirty-second Annual Meeting, Columbus, Ohio, December 31, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pennsylvania State College, May 8, 1948

ILLINOIS, Illinois Institute of Technology, Chicago, May 14–15, 1948

INDIANA, Purdue University, West Lafayette, May 8, 1948

IOWA, Fairfield, April 16–17, 1948

KANSAS, Atchison, April 10, 1948

KENTUCKY, Berea, May, 1948

LOUISIANA-MISSISSIPPI, Southwestern Louisiana Institute, Lafayette, La., April 23–24, 1948

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, United States Naval Academy, Annapolis, Maryland, May 8, 1948

METROPOLITAN NEW YORK, Washington Irving High School, April 24, 1948

MICHIGAN, University of Michigan, Ann Arbor, April 3, 1948

MINNESOTA, College of St. Thomas, St. Paul, May 8, 1948

MISSOURI, University of Kansas City, Kansas City, April 23, 1948

NEBRASKA, University of Nebraska, Lincoln, May 1, 1948

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 3, 1948

OKLAHOMA

PACIFIC NORTHWEST, Eugene, Oregon, March 26–27, 1948

PHILADELPHIA, Philadelphia, Pa., Nov. 27, 1948

ROCKY MOUNTAIN, April 23–24, 1948

SOUTHEASTERN, The Citadel, Charleston, South Carolina, March 19–20, 1948

SOUTHERN CALIFORNIA, Redlands, March 13, 1948

SOUTHWESTERN, New Mexico Highlands University, Las Vegas, New Mexico, May 3–6, 1948

TEXAS, Rice Institute, Houston, April 23–24, 1948

UPPER NEW YORK STATE, Schenectady, May 1, 1948

WISCONSIN, Beloit, May 8, 1948

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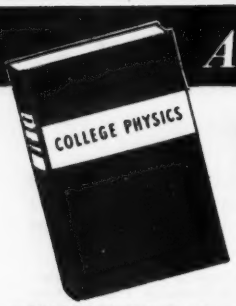

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